POSITIVE-DEFINITE MATRICES AND THEIR APPLICATIONS TO THE PROBLEMS OF INTERNATIONAL MATHEMATICAL OLYMPIADS

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Annotation: *This article reflected on positive and negative defined matrices. Suitable definitions were given for such matrices. Theorem about the general appearance positive-definite matrices is given with its proof. The connection between a Gram Matrix and a positively defined matrix is also given. Solutions to complex Olympic problems are given. At the end of the article, enough problems are given to work independently.*

Keywords*:* *symmetric real matrix, Hermitan matrix, positive-definite, positive-semidefinite, negative-definite, negative-semidefinite, Gauss' Theorem, Gram matrix, Euclidean space, diagonalizable, skew-symmetric matrix, eigenvalues*

1. INTRODUCTION

Symmetric matrix M with real entries is ***positive-definite*** if the real number $x^T Mx$ is positive for every nonzero real column vector x, where x^T is the row vector transpose of x. More generally, a ***Hermitan matrix*** (that is, a complex matrix equal to its conjugate transpose) is ***positive-definite*** if the real number $z^* Mz$ is positive for every nonzero complex column vector z, where z^* denotes the conjugate transpose of z.

Positive semi-definite matrices are defined similarly, except that the scalars $x^T Mx$ and $z^* Mz$ are required to be *positive or zero* (that is, non-negative). ***Negative-definite*** and ***negative semi-definite*** matrices are defined analogously. A matrix that is not positive semi-definite and not negative semi-definite is sometimes called ***indefinite.***

Positive (negative) defined matrices have a number of nice properties. We need to define these properties as well as include a complete definition before we can use it in solving the problem.

2. Definitions for positive and negative matrix.

Let us give detailed definitions for positive (negative) defined matrices.

Definition 2.1. An $n \times n$ symmetric real matrix M is said to be ***positive-definite*** if $x^T Mx > 0$ for all non-zero $\boldsymbol{x}$ in $\mathbb{R}^n$. Formally,

$$M \text{ positive-definite } \iff x^T Mx > 0 \text{ for all } \boldsymbol{x} \in \mathbb{R}^n \backslash \{0\}.$$

Definition 2.2. An $n \times n$ symmetric real matrix M is said to be ***positive-semidefinite*** or ***non-negative-definite*** if $x^T Mx \geq 0$ for all $\boldsymbol{x}$ in $\mathbb{R}^n$. Formally,

$$M \text{ positive-semidefinite } \iff x^T Mx \geq 0 \text{ for all } \boldsymbol{x} \in \mathbb{R}^n.$$

Definition 2.3. An $n \times n$ symmetric real matrix M is said to be ***negative-definite*** if $x^T Mx < 0$ for all non-zero $\boldsymbol{x}$ in $\mathbb{R}^n$. Formally,

$$M \text{ negative-definite } \iff x^T Mx < 0 \text{ for all } \boldsymbol{x} \in \mathbb{R}^n \backslash \{0\}.$$

Definition 2.4. An $n \times n$ symmetric real matrix M is said to be ***negative-semidefinite*** or ***non-positive-definite*** if $x^T Mx \leq 0$ for all $\boldsymbol{x}$ in $\mathbb{R}^n$. Formally,

$$M \text{ negative-semidefinite } \iff x^T Mx \leq 0 \text{ for all } \boldsymbol{x} \in \mathbb{R}^n.$$

In other words, $\boldsymbol{M} = [\boldsymbol{a}_{ij}]$ is non-negative (respectively positive definite) if and only if the quadratic form associated with $\boldsymbol{M}$, namely $(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_n) \to \sum_{i,j=1}^{n} \boldsymbol{a}_{ij}\boldsymbol{x}_i\boldsymbol{x}_j$, is non-negative (respectively positive definite).

Note that if $\boldsymbol{M}$ is non-negative then letting $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ be the canonical basis of $\mathbb{R}^n$ we have

$$a_{ii} = e_i^T M e_i \geq 0,$$

and if M is positive definite then the inequality is strict. Also, note that any matrix congruent to a **non-negative** (respectively positive definite) symmetric matrix is itself **non-negative** (respectively positive definite), since

$$X^T(P^T MP)X = (PX)^T M(PX).$$

3. Theorem for positive definite matrices.

By our theorem given in this section, we define the general representation of positively defined matrices! Before giving the theorem, let's consider the following problem.

Problem 3.1. **Let $\boldsymbol{A} \in \boldsymbol{M}_n(\mathbb{R})$ any matrix.**

$\boldsymbol{a}$**) Prove that $\boldsymbol{A}^T\boldsymbol{A}$ is symmetric and non-negative.**

$\boldsymbol{b}$**) Prove that $\boldsymbol{A}^T\boldsymbol{A}$ is positive definite if and only if $\boldsymbol{A}$ is invertible.**

$$(A^T A)^T = A^T (A^T)^T = A^T A,$$

thus **$\boldsymbol{A}^T\boldsymbol{A}$ is symmetric. Next, for all $\boldsymbol{x} \in \mathbb{R}^n$ we have**

$$x^T(A^T A)x = (Ax)^T(Ax) = ||Ax||^2 \geq 0,$$

with equality if and only if $Ax = 0$. Both $a)$ and $b)$ follow from these observations (and the fact that A is invertible if and only if $Ax = 0$ implies $x = 0$).

We have the same result holds with AA^T instead of A^TA. Remarkably, the converse of the result established in the previous problem holds:

Theorem 3.1. Any non-negative-definite matrices $A \in M_n(\mathbb{R})$ can be written as B^TB for some matrix $B \in M_n(\mathbb{R})$.

Proof: **We use Gauss' Theorem (see [1], p 402.). By that theorem, there is an invertible matrix P such that $P^TAP = D$ is a diagonal matrix. By the discussion preceding** Problem 3.1 **we know that D itself is non-negative and its diagonal coefficients d_{ii} are nonnegative. Hence we can write $D = D_1^TD_1$ for a diagonal matrix D_1 whose diagonal entries are $\sqrt{d_{ii}}$. But then**

$$A = (P^{-1})^TDP^{-1} = (P^{-1})^TD_1^TD_1P^{-1} = B^TB,$$

with $B = D_1P^{-1}$. Done!

In the following problem, we show that the *Gram matrix* **is non-negative defined. And we also give a necessary and sufficient condition for being positive-defined.**

Problem 3.2. **Let $(V, \langle,\rangle)$ be an Euclidean space and let $v_1, v_2, \dots, v_n$ be a family of vectors in V. Let $A \in M_n(\mathbb{R})$ be the** *Gram matrix* **of the family, i.e., the matrix whose (i,j) − entry is $\langle v_i, v_j\rangle$.**

$a)$ Prove that A is symmetric positive.

$b)$ Prove that A is positive definite if and only if $v_1, v_2, \dots, v_n$ are linearly independent.

$$\sum_{i,j=1} a_{ij}x_ix_j = \sum_{i,j=1} x_ix_j\langle v_i, v_j\rangle = \sum_{i=1} x_i \cdot \sum_{j=1}\langle v_i, x_jv_j\rangle =$$

$$= \sum_{i=1}^{n}\langle x_iv_i, \sum_{j=1}^{n} x_jv_j\rangle = \left\|\sum_{i=1}^{n} x_iv_i\right\|^2 \geq 0,$$

with equality if and only if $\sum_{i=1}^{n} x_iv_i = 0$. The result follows.

Let's analyze another interesting problem below.

Problem 3.3. Let $n \geq 1$ and let $A = [a_{ij}] \in M_n(\mathbb{R})$ be defined by $a_{ij} = \min(i,j)$. Prove that A is symmetric and positive definite.

Solution: It is clear that the matrix is symmetric. Note that we can write

$$\min(i,j) = \sum_{k\leq i, k\leq j} 1.$$

Doing so and interchanging orders of summation, we see that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\min(i,j)\,x_i x_j=\sum_{k=1}^{n}\sum_{i=k}^{n}\sum_{j=k}^{n}x_i x_j=\sum_{k=1}^{n}\left(\sum_{i=k}^{n}x_i\right)^2.$$

This last expression is clearly nonnegative, and it equals 0 if and only if

$$x_1+\cdots+x_n=0, x_2+\cdots+x_n=0,\ldots,x_n=0.$$

Subtracting each equation from the one before it, we see that the unique solution is $x_1=x_2=\cdots=x_n=0$, which shows that the matrix is positive definite.

An alternative argument is to note that

$$\min(i,j)=\int_0^\infty f_i(x)f_j(x)dx,$$

where $f_i(x)=1$ for $x\in[0,i]$ and $f_i(x)=0$ for $x>i$ (i.e., f_i is the characteristic function of the interval $[0,i]$). It follows that A is the Gram matrix of the family $f_1,f_2,\ldots,f_n$, thus it is symmetric and non-negative. Since $f_1,f_2,\ldots,f_n$ are linear independent (in the space of integrable functions on $[0,\infty)$), it follows that A is positive definite (all this uses **Problem 3.2**).

4. Application of the *Positive-definite matrices.*

4.1. Examples of International Olympic problems.

Example 4.1.1 (OMOUS-2024, Turkmenistan).

Show that if A is a real $n\times n$ matrix such that $A+A^2A^T+(A^2)^T=0$, then $A=0$.

$$A+A^2A^T+(A^2)^T=0\,|\cdot A^T$$
$$AA^T+A^2(A^2)^T+(A^3)^T=0$$
$$(A^3)^T=-(AA^T+A^2(A^2)^T)\le 0.$$

Hence $I-A^3$ is positive definite and invertible.

On the other hand we have

$$(A^2)^T=-A-A^2A^T$$

and transposing we get

$$A^2=-A^T-A(A^2)^T.$$

Hence we get

$$A^2=-A^T+A(A+A^2A^T).$$

Thus $A^T=A^3A^T$, so $A^T(I-A^3)=0$. This together with invertibility of $I-A^3$ implies that $A=0$.

Example 4.1.2 (RUDN MATH OLYMP-2024, Russia).

Let $A,B\in M_n(\mathbb{R})$ be symmetric martices such that

$$(AB+BA-A^2-I_n)^2=AB^2-B^2A.$$

Find $rank(A^2+B^2)$

$$(AB^2 - B^2A)^T = (AB^2)^T - (B^2A)^T = (B^T)^2A^T - A^T(B^T)^2 = -(AB^2 - B^2A).$$

We defined $M = (AB + BA - A^2 - I_n)$.

$$M^T = (AB + BA - A^2 - I_n)^T = (B^TA^T + A^TB^T - (A^T)^2 - I_n) =$$
$$= BA + AB - A^2 - I_n = M.$$

Hence,

$$-(AB^2 - B^2A) = (AB^2 - B^2A)^T = (M^2)^T = (M^T)^2 = M^2 = AB^2 - B^2A$$
$$AB^2 - B^2A = O_n \Rightarrow M^2 = O_n$$

M is symmetric, then it is diagonalizable. Then $\exists C \in M_n(\mathbb{R})$, $\det C \neq 0$ and

$$CMC^{-1} = diag(\lambda_1, \lambda_2, \dots, \lambda_n)$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues of M and they are real numbers.

$$O_n = M^2 = CM^2C^{-1} = (CMC^{-1})^2 = diag(\lambda_1^2, \dots, \lambda_n^2)$$

Hence, $\lambda_i = 0$ for all $i = 1,2, \dots, n$. Therefore,

$$CMC^{-1} = O_n \Rightarrow M = O_n \Rightarrow$$
$$BA + AB - A^2 - I_n = O_n \ (*)$$

We know that $A^2 = A^TA$ and $B^2 = B^TB$, so $A^2 + B^2$ is non-negative-definite (since **Problem 3.1**). Hence for all $x \in \mathbb{R}^n$,

$$x^T(A^2 + B^2)x \geq 0 \Rightarrow (Ax)^T(Ax) + (Bx)^T(Bx) = ||Ax||^2 + ||Bx||^2 \geq 0.$$

Assume that $\exists x \in \mathbb{R}^n \backslash \{\theta\}$ such that $(A^2 + B^2)x = \theta$, where $\theta = (0, \dots, 0)$. Then, $Ax = Bx = \theta$. Since $(*)$ we have that,

$$(BA + AB - A^2 - I_n)x = O_nx$$
$$BAx + ABx - A^2x - x = \theta \Rightarrow x = \theta.$$

This is contradiction! Hence, $(A^2 + B^2)x = \theta$ if and only if $x = \theta$. Therefore $\det(A^2 + B^2) > 0$. Then $rank(A^2 + B^2) = n$. ***Answer:*** $rank(A^2 + B^2) = n$.

Example 4.1.3. (AKHIMO-2023, Uzbekistan).

Let $A, B, C \in M_n(\mathbb{R})$ be skew-symmetric $(M^T = -M)$ matrices such that:

$$\det(A^2 + B^2 + C^2) = 0.$$

Prove that for each triple $P, Q, R \in M_n(\mathbb{R})$ the following equality holds

$$\det(AP + BQ + CR) = 0$$

$$-(A^TA + B^TB + C^TC)x = \theta.$$

Multiply both sides x^T to get

$$x^T(A^TA + B^TB + C^TC)x = \theta$$
$$(Ax)^T(Ax) + (Bx)^T(Bx) + (Cx)^T(Cx) = \theta.$$

Notice that $(Ax)^T(Ax)$ is non-negative-definite. Thus $Ax = Bx = Cx = \theta$.

Therefore,

$$\theta = Ax = x^TA^T = -x^TA = \theta.$$

This implies that

$$x^T(AP + BQ + CR) = \theta.$$

We know that $x^T \neq \theta^T$. Finally, we conclude that

$$det(AP + BQ + CR) = 0.$$

4.2. Problems for Practice.

Problem 4.2.1. Is the matrix $A = [a_{ij}] \in M_n(\mathbb{R})$ with $a_{ij} = i \cdot j$ non-negative-definite? Is it positive definite?

Problem 4.2.2.

$a)$ Prove that a symmetric positive definite matrix is invertible.

$b)$ Prove that a symmetric positive matrix is positive definite if and only if it is invertible.

Problem 4.2.3. Prove that any symmetric positive matrix $A \in M_n(\mathbb{R})$ is the Gram matrix of a family of vectors $v_1, v_2, \dots, v_n \in \mathbb{R}^n$.

Problem 4.2.4. Prove that the matrix $A = \left[\frac{1}{i+j}\right]_{1 \le i,j \le n}$ is symmetric and non-negative-definite.

Problem 4.2.5. Let $A = [a_{ij}] \in M_n(\mathbb{R})$ be a matrix such that $a_{ij} = 1$ for $i \neq j$, and $a_{ii} > 1$ for all $i = 1,2,\dots,n$. Prove that A is symmetric and non-negative-definite.

Problem 4.2.6 (SEEMOUS-2024, Romania). Let $A, B \in M_n(\mathbb{R})$ two real, symmetric matrices with nonnegative eigenvalues.

Prove that $A^3 + B^3 = (A + B)^3$ if and only if $AB = O_n$.

Problem 4.2.7 (SEEMOUS-2024, Romania). Let $A \in M_n(\mathbb{C})$ be a matrix such that $(AA^*)^2 = AA^*$, where $A^* = (\bar{A})^T$ denotes the Hermitian transpose (i.e., the conjugate transpose) of A.
$a)$ Prove that $AA^* = A^*A$.
$b)$ Show that the non-zero eigenvalues of A have modulus one.

REFERENCES :

[1] Titu Andreescu, Essential Linear Algebra with Applications, A Problem-Solving Approach, Springer Science+Business Media New York, 2014.

[2] MV Alba-Fernández, Apostolos Batsidis, María-Dolores Jiménez-Gamero, and Pedro Jodrá. A class of tests for the two-sample problem for count data. Journal of Computational and Applied Mathematics, 318:220–229, 2017.

[3] Gustav Alfelt, Taras Bodnar, and Joanna Tyrcha. Goodness-of-fit tests for centralized Wishart processes. Communications in Statistics-Theory and Methods, 49(20):5060–5090, 2020.

[4] Shanshan Ding and R Dennis Cook. Matrix variate regressions and envelope models. Journal of the Royal Statistical Society Series B: Statistical Methodology, 80(2):387–408, 2018.

[5] Xu B, Xiao M Ch, Zhou J R, （2008）Analysis of Exam Questions and Countermeasures for Graduate School Entrance Examination, Higher Education Press, Bei jing

[6] Department of Mathematics, Tongji University, (2019) Linear Algebra, 6th edition, Higher Education Press, Bei jing .

[7] Wan Y, Zhang H, "Proofing Method of Positive Definite Matrix", Journal of Jiamusi Vocational College, No. 8, 2015, pp .286. [5] Waclaw Sierpiński. Elementary Theory of Numbers. PWN, Warsaw, 1964.

[8] Tang J F,(2016), Lecture Notes for Linear Algebra, China Atomic Energy Press.

[9] Wolfgang Fruehwirt, Leonhard Hochfilzer, Leonard Weydemann, and Stephen Roberts. Cumulation, crash, coherency: A cryptocurrency bubble wavelet analysis. Finance Research Letters, 40:101668, 2021.

[10] Marija Cuparic, Bojana Miloševi ´ c, and Marko Obradovi ´ c. New consistent exponentiality tests based on V- ´ empirical Laplace transforms with comparison of efficiencies. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 116(1):42, 2022.

[11] Markus Bibinger, Nikolaus Hautsch, Peter Malec, and Markus Reiss. Estimating the spot covariation of asset prices—statistical theory and empirical evidence. Journal of Business & Economic Statistics, 37(3):419–435, 2019.

[12] www.aops.com internet sayti.

[13] https://math.stackexchange.com/ internet sayti.

[14] http://www.imc-math.uk/ internet sayti.

[15] https://math-olymp.rudn.ru/ internet sayti.

[16] https://iuhd.edu.tm/competition/83 internet sahifasi.

[17] https://www.researchgate.net internet sayti.