# CALCULATING SOME INTEGRALS IN MATRIX DOMAINS

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**Annotation:** *In this article, some complex integrals in mathematical analysis are solved using Euler integrals.*

**Keys:** *Integral, Euler integral, Gamma function, symmetric matrix.*

**Annotatsiya:** *Bu maqolada matematik analizdagi bazi bir murakkab integrallar eyler integrallari yordamida hisoblangan.*

**Kalit so'zlar:** *Integral, Eyler integrali, Gamma funksiya, simmetrik matritsa*

**Аннотация:** *В данном статья рассматриваются способы решения некоторых сложных интегралов математического анализа с использованием интегралов Эйлера.*

**Ключевые слово:** *Интеграл, интеграл Эйлера, гамма-функция, симметричная матрица.*

In the course of mathematical analysis, we widely use the methods of calculating definite integrals when calculating improper integrals. However, we encounter several problems when calculating some integrals with these methods.

In such cases, using Euler integrals gives a good result. We will consider this in the following two examples.

Theorem 1. If $\alpha > \frac{n}{2}$, then

$$I_n(\alpha) = \int_T \frac{T}{(\det(I+T^2))^\alpha} = 2^{\frac{n(n-1)}{4}} \pi^{\frac{n(n+1)}{4}} \frac{\Gamma(\alpha - \frac{n}{2})}{\Gamma(\alpha)} \prod_{\nu=1}^{n-1} \frac{\Gamma(2\alpha - \frac{n+\nu}{2})}{\Gamma(2\alpha - \nu)}.$$

Here $T = (t_{jk})_1^n$ passes through all real symmetric matrices of the n-th order, and $T = 2^{\frac{n(n-1)}{4}} \prod_{j \le k} dt_{jk.}$.

Example 1. Given a parametric view

$$\int_{-\infty}^{+\infty} \frac{1}{(1+x^2)^\alpha} dx \quad (\alpha > 0)$$

calculate the integral.

Solving: First we calculate the $x = tg\left(\arcsin\sqrt{y}\right)$ permutation,

$$\int_{-\infty}^{+\infty} \frac{1}{(1+x^2)^\alpha} dx = 2\int_0^1 \frac{1}{\left(1+tg^2\left(\arcsin\sqrt{y}\right)\right)^\alpha} \cdot \frac{1}{2\sqrt{y}(1-y)^{\frac{3}{2}}} dy = \int_0^1 y^{\frac{1}{2}-1}(1-y)^{\left(\alpha-\frac{1}{2}\right)-1} dy \qquad (1)$$

we will have equality. This equation can be written in

$$B(1/2, \alpha - 1/2) = \int_0^1 y^{\frac{1}{2}-1}(1-y)^{\left(\alpha-\frac{1}{2}\right)-1} dy$$

using Euler's integral of the first kind (beta function) and using Euler's integral of the second kind (gamma function) ([1,2]) we get the following result:

$$B(1/2,\alpha-1/2)=\int_0^1 y^{\frac{1}{2}-1}(1-y)^{\left(\alpha-\frac{1}{2}\right)-1}dy=\frac{\Gamma(1/2)\Gamma(\alpha-1/2)}{\Gamma(\alpha)}=\frac{\sqrt{\pi}\Gamma(\alpha-1/2)}{\Gamma(\alpha)}$$

In the special case when $\alpha=1$: $\int_{-\infty}^{+\infty}\frac{1}{1+x^2}dx=\pi$

We will first prove the following two theorems.

Theorem 2. Let Z - m× n-matrix (i.e., matrix consisting of n rows and n columns). Then

$$\det(I^{(m)}-Z\bar{Z}')=\det(I^{(n)}-\bar{Z}'Z).$$

Moreover, the conditions $I^{(m)}-Z\bar{Z}'>0$ and $I^{(n)}-\bar{Z}'Z>0$ are equivalent ( $I^{(m)}$ - a unix matrix of order m).

Proof. It is well known that any m × n −matrix Z can be represented as

Z=UΛV

Where U and V are unitary matrices of the order m and n respectively, and the m × n −matrix Λ has the form

$$\Lambda=\begin{pmatrix}\lambda_1 & 0\ldots & 0\\ 0 & \lambda_2\ldots & 0\\ \ldots & \ldots & \ldots\end{pmatrix},\ \lambda_n\ge 0$$

From this it follows

$$\det(I^{(m)}-Z\bar{Z}')=(1-\lambda_1^2)(1-\lambda_2^2)\ldots=\det(I^{(n)}-\bar{Z}'Z).$$

The second statement of the theorem is obtained from the consideration that both mentioned conditions are equivalent to the condition $|\lambda_1|<1, |\lambda_2|<1, \ldots$ .

Theorem 3. a>0, $\int_{-\infty}^{\infty}\frac{1}{(ax^2+2bx+c)^\alpha}dx$. $a>0, b^2-4ac<0, \alpha>\frac{1}{2}$, then

$$\int_{-\infty}^{\infty}\frac{1}{(ax^2+2bx+c)^\alpha}dx=\frac{1}{\sqrt{a}}\left(\frac{a}{ac-b^2}\right)^{\alpha-\frac{1}{2}}\cdot\frac{\sqrt{\pi}\Gamma(\alpha-\frac{1}{2})}{\Gamma(\alpha)}$$

Proof: First, we write the square root of $ax^2+2bx+c$ in perfect square form, giving us the equation

$$\int_{-\infty}^{\infty}\frac{1}{(ax^2+2bx+c)^\alpha}dx=\left(\frac{a}{ac-b^2}\right)^\alpha\int_{-\infty}^{\infty}\frac{dx}{\left(1+\frac{a^2}{ac-b^2}(x+\frac{b}{a})^2\right)^\alpha}. \quad (2)$$

This improper integral has the form of the previous integral. In this integral, we also get the result below by trying to use $\frac{a}{\sqrt{ac-b^2}}(x+\frac{b}{a})=tg(\arcsin\sqrt{y})$.

$$(\frac{a}{ac-b^2})^{\alpha}\frac{\sqrt{ac-b^2}}{a}\int_0^1 y^{\frac{1}{2}-1}(1-y)^{(\alpha-\frac{1}{2})-1}dy=\frac{1}{\sqrt{a}}(\frac{a}{ac-b^2})^{\alpha-\frac{1}{2}}\cdot\frac{\sqrt{\pi}\Gamma(\alpha-\frac{1}{2})}{\Gamma(\alpha)}$$

$$\int_{-\infty}^{\infty}\frac{1}{(ax^2+2bx+c)^{\alpha}}dx=(\frac{a}{ac-b^2})^{\alpha}\int_{-\infty}^{\infty}\frac{dx}{(1+\frac{a^2}{ac-b^2}(x+\frac{b}{a})^2)^{\alpha}}=$$

$$(\frac{a}{ac-b^2})^{\alpha}\frac{\sqrt{ac-b^2}}{a}\int_0^1 y^{\frac{1}{2}-1}(1-y)^{(\alpha-\frac{1}{2})-1}dy=\frac{1}{\sqrt{a}}(\frac{a}{ac-b^2})^{\alpha-\frac{1}{2}}\cdot\frac{\sqrt{\pi}\Gamma(\alpha-\frac{1}{2})}{\Gamma(\alpha)}.$$

Proof of theorem 2. Let

$$T=\begin{pmatrix}T_1 & \upsilon' \\ \upsilon & t\end{pmatrix}\quad (t=t_{nn}),$$

where $T_1$ is a real symmetric matrix of order n-1, $\upsilon$ – n-1) – dimensional vector, t – is a real number. Then

$$I+T^2=\begin{pmatrix}I+T_1^2+\upsilon'\upsilon & T_1\upsilon'+\upsilon't \\ \upsilon T_1+t\upsilon & 1+\upsilon\upsilon'+t^2\end{pmatrix}.$$

Since at $A=A'$

$$\begin{pmatrix}I & 0 \\ -bA^{-1} & 1\end{pmatrix}\begin{pmatrix}A & b' \\ b & c\end{pmatrix}\begin{pmatrix}I & 0 \\ -bA^{-1} & 1\end{pmatrix}'=\begin{pmatrix}A & 0 \\ 0 & c-bA^{-1}b'\end{pmatrix}.$$

Then

$$\det(I+T^2)=\det(I+T_1^2+\upsilon'\upsilon)\times\{1+\upsilon\upsilon'+t^2-(\upsilon T_1+t\upsilon)(I+T_1^2+\upsilon'\upsilon)^{-1}(T_1\upsilon'+\upsilon't)\}.$$

The second factor on the right side of this equality can be written as $at^2+2bt+c$, where

$$a=1-\upsilon(I+T_1^2+\upsilon'\upsilon)^{-1}\upsilon',$$

$$2b=-\upsilon T_1(I+T_1^2+\upsilon'\upsilon)^{-1}\upsilon'-\upsilon(I+T_1^2+\upsilon'\upsilon)^{-1}T_1\upsilon'=-2\upsilon(I+T_1^2+\upsilon'\upsilon)^{-1}T_1\upsilon',$$

$$c=1+\upsilon\upsilon'-\upsilon T_1(I+T_1^2+\upsilon'\upsilon)^{-1}T_1\upsilon'.$$

But the symmetric matrix $T_1$ can be represented as

$$T_1=\Gamma'[\lambda_1,\ldots,\lambda_{n-1}]\Gamma.$$

Where $\Gamma$ is some orthogonal matrix.

Lets put

$$T_2=\Gamma'[\sqrt{1+\lambda_1^2},\ldots,\sqrt{1+\lambda_{n-1}^2}]\Gamma.$$

Then

$$T_2=T_2',\quad T_1T_2=T_2T_1,\quad I+T_1^2=T_2^2.$$

If we put about $\upsilon=\omega T_2$, then we get

$$\upsilon=\det T_2\cdot\omega=(\det(I+T_1^2))^{\frac{1}{2}}\cdot\omega$$

and

$$I+T_1^2+\upsilon'\upsilon=T_2(I+\omega'\omega)T_2.$$

Moreover, if u - is an (n-1) – dimensional vector, then [by virtue of the equality $(\omega'\omega)^2=\omega'\omega(\omega\omega')$] we have

$$u(I+\omega'\omega)^{-1}u'=uu'-\frac{(u\omega')^2}{1+\omega\omega'}$$

and

$$\omega(I+\omega'\omega)^{-1}=\frac{\omega}{1+\omega\omega'}.$$

Means,

$$a=1-\omega(I+\omega'\omega)^{-1}\omega'=\frac{1}{1+\omega\omega'},$$

$$b=-\omega(I+\omega'\omega)^{-1}T_1\omega'=-\frac{\omega T_1\omega'}{1+\omega\omega'},$$

$$c=1+\omega T_2^2\omega'-\omega T_1(I+\omega'\omega)^{-1}T_1\omega'=1+\omega\omega'+\frac{(\omega T_1\omega')^2}{1+\omega\omega'}.$$

Hence,

$$ac-b^2=1.$$

By theorem 3 we have

$$I_n(\alpha)=\int_T\frac{T}{[\det(I+T^2)]^\alpha}=2^{\frac{n-1}{2}}\int_{t,\upsilon,T_1}[\det(I+T_1^2+\upsilon'\upsilon)]^{-\alpha}(at^2+2bt+c)^{-\alpha}dt\upsilon T_1=$$

$$2^{\frac{n-1}{2}}\sqrt{\pi}\frac{\Gamma(\alpha-\frac{1}{2})}{\Gamma(\alpha)}\int_\omega(1+\omega\omega')^{1-2\alpha}\omega\int_{T_1}[\det(I+T_1^2)]^{\frac{1}{2}-\alpha}T_1.$$

Example 3. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1+x^2+y^2\right)^{1-2\alpha}dxdy$, α>1. Calculate the integral.

Solving. First, we perform the following calculation.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1+x^2+y^2\right)^{1-2\alpha}dxdy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\left(1+y^2+x^2\right)^{2\alpha-1}}dxdy=\int_{-\infty}^{\infty}\left(1+y^2\right)^{1-2\alpha}\int_{-\infty}^{\infty}\frac{1}{\left(1+\frac{x^2}{1+y^2}\right)^{2\alpha-1}}dxdy$$

(3)

This improper integral has the form of the previous integral. In this integral, we also get the result by trying to use $\frac{x}{\sqrt{1+y^2}}=tg\left(\arcsin\sqrt{A}\right)$. As a result, we obtain the following expression.

$$\int_{-\infty}^{\infty}\left(1+y^2\right)^{1-2\alpha}\int_{-\infty}^{\infty}\frac{1}{\left(1+\frac{x^2}{1+y^2}\right)^{2\alpha-1}}dxdy=\int_{-\infty}^{\infty}\left(1+y^2\right)^{\frac{3}{2}-2\alpha}\int_0^1A^{\frac{1}{2}-1}\left(1-A\right)^{2\alpha-\frac{3}{2}-1}dAdy=$$

$$=\mathrm{B}\left(\frac{1}{2},2\alpha-\frac{3}{2}\right)\int_{-\infty}^{\infty}\left(1+y^2\right)^{\frac{3}{2}-2\alpha}dy.$$

$$\int_{-\infty}^{\infty}\left(1+y^2\right)^{\frac{3}{2}-2\alpha}dy=\int_{-\infty}^{\infty}\frac{1}{\left(1+y^2\right)^{2\alpha-\frac{3}{2}}}dy$$

Then we enter the notation $y=tg\left(\arcsin A_2\right)$ to calculate the following integral.

$$\int_{-\infty}^{\infty}\frac{1}{\left(1+y^2\right)^{2\alpha-\frac{3}{2}}}dy=\int_{0}^{1}A_2^{\frac{1}{2}-1}\left(1-A_2\right)^{2\alpha-3}=\mathrm{B}\left(\frac{1}{2},2\alpha-2\right).$$

$$\mathrm{B}\left(\frac{1}{2},2\alpha-\frac{3}{2}\right)\mathrm{B}\left(\frac{1}{2},2\alpha-2\right)=\pi\frac{\Gamma\left(2\alpha-2\right)}{\Gamma\left(2\alpha-1\right)}.$$

Now from the known formula ([3,5]) $\left(\alpha>\frac{n+1}{4}\right)$

$$\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}\left(1+x_1^2+\ldots+x_{n-1}^2\right)^{1-2\alpha}dx_1\ldots dx_{n-1}=\pi^{\frac{n-1}{2}}\frac{\Gamma\left(2\alpha-\frac{n+1}{2}\right)}{\Gamma\left(2\alpha-1\right)},$$

we obtain a reccurence relation

$$I_n(\alpha)=2^{\frac{n-1}{2}}\pi^{\frac{n}{2}}\frac{\Gamma\left(2\alpha-\frac{n+1}{2}\right)\Gamma\left(\alpha-\frac{1}{2}\right)}{\Gamma(\alpha)\Gamma\left(2\alpha-1\right)}I_{n-1}\left(\alpha-\frac{1}{2}\right).$$

Applying the recurrence formula n - 1 times and noting that

$$I_1\left(\alpha-\frac{n-1}{2}\right)=\int_{-\infty}^{\infty}\frac{dx}{\left(1+x^2\right)^{\alpha-\frac{n-1}{2}}}=\frac{\sqrt{\pi}\,\Gamma\left(\alpha-\frac{n}{2}\right)}{\Gamma\left(\alpha-\frac{n-1}{2}\right)}\qquad\alpha>\frac{n}{2},$$

we get our approval.

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