RADIAL STABILITY OF SPHERICALLY SYMMETRIC BLACK HOLES IN SCALAR-TENSOR GRAVITY COUPLED TO NONLINEAR ELECTRODYNAMICS

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Abstract: *We investigate the radial stability of asymptotically flat, spherically symmetric black hole solutions in scalar-tensor gravity coupled to nonlinear electrodynamics (NED). The inclusion of the NED sector bypasses standard scalar no-hair theorems, providing a rigorous mechanism for the existence of horizons with non-trivial scalar profiles. By considering time-dependent monopole ($l = 0$) perturbations, we derive the fully coupled linearized equations governing the gravitational and scalar fields. Recognizing that electromagnetic perturbations do not introduce independent dynamical degrees of freedom in the monopole sector, we utilize a gauge-invariant framework to decouple the system into a single master wave equation governed by a scalar effective potential. The stability of the static background is analyzed through the properties of this effective potential. In parameter regions where the potential exhibits negative wells, we strictly apply the S-deformation method to rule out the existence of unstable normalizable bound states. Our analytical results establish the exact parametric bounds required for the radial stability of these configurations, demonstrating that stability is rigidly constrained by the NED coupling parameter and the effective scalar charge.*

Keywords: *Scalar-tensor gravity, Nonlinear electrodynamics, Black hole stability, Radial perturbations, S-deformation method.*

РАДИАЛЬНАЯ УСТОЙЧИВОСТЬ СФЕРИЧЕСКИ-СИММЕТРИЧНЫХ ЧЕРНЫХ ДЫР В СКАЛЯРНО-ТЕНЗОРНОЙ ГРАВИТАЦИИ, ВЗАИМОДЕЙСТВУЮЩЕЙ С НЕЛИНЕЙНОЙ ЭЛЕКТРОДИНАМИКОЙ

Аннотация: *Мы исследуем радиальную устойчивость асимптотически плоских, сферически-симметричных решений для черных дыр в скалярно-тензорной теории гравитации, взаимодействующей с нелинейной электродинамикой (НЭД). Включение сектора НЭД позволяет обойти стандартные теоремы об «отсутствии скалярных волос», обеспечивая строгий механизм существования горизонтов с нетривиальными скалярными профилями. Рассматривая зависящие от времени монопольные ($l = 0$) возмущения, мы выводим полностью связанные линеаризованные уравнения,*

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описывающие гравитационное и скалярное поля. Учитывая, что электромагнитные возмущения не порождают независимых динамических степеней свободы в монопольном секторе, мы используем калибровочно-инвариантный формализм для сведения системы к единственному основному волновому уравнению (master equation), определяемому скалярным эффективным потенциалом. Устойчивость статического фона анализируется на основе свойств данного эффективного потенциала. В тех областях параметров, где потенциал формирует отрицательные ямы, мы применяем метод S-деформации для исключения возможности существования неустойчивых нормируемых связанных состояний. Наши аналитические результаты устанавливают точные параметрические границы, необходимые для радиальной устойчивости данных конфигураций, демонстрируя, что устойчивость жестко ограничивается параметром связи НЭД и эффективным скалярным зарядом.

Ключевые слова: *Скалярно-тензорная гравитация, Нелинейная электродинамика, Устойчивость черных дыр, Радиальные возмущения, Метод S-деформации.*

NOCHIZIQLI ELEKTRODINAMIKA BILAN TA’SIRLASHUVCHI SKALYAR-TENZOR GRAVITATSIYASIDA SFERIK-SIMMETRIK QORA TUYNUKLARNING RADIAL BARQARORLIGI

Annotatsiya: *Biz nochiziqli elektrodinamika (NED) bilan bog'langan skalyar-tenzor gravitatsiyasidagi asimptotik tekis, sferik simmetrik qora tuynuk yechimlarining radial barqarorligini tadqiq etamiz. NED sektorining kiritilishi standart skalyar "sochsizlik" (no-hair) teoremalarini aylanib o'tishga imkon beradi hamda notestik (notrivial) skalyar profilli ufqitlarning mavjudligi uchun qat'iy mexanizmni ta'minlaydi. Vaqtga bog'liq monopol (l=0) g'alayonlanishlarni ko'rib chiqish orqali biz gravitatsion va skalyar maydonlarni boshqaruvchi to'liq o'zaro bog'langan chiziqli tenglamalarni keltirib chiqaramiz. Elektromagnit g'alayonlanishlar monopol sektorida mustaqil dinamik erkinlik darajalarini keltirib chiqarmasligini e'tiborga olib, biz tizimni skalyar samarali potensial boshqaradigan yagona asosiy to'lqin tenglamasiga ajratish uchun kalibrovka-invariant yondashuvdan foydalanamiz. Statik fonning barqarorligi ushbu samarali potensialning xossalari orqali tahlil qilinadi. Potensial manfiy o'ralarni namoyon qiluvchi parametrlar sohalarida biz beqaror normallashuvchi bog'langan holatlarning mavjudligini qat'iy ravishda istisno qilish uchun S-deformatsiya usulini tatbiq etamiz. Bizning analitik natijalarimiz ushbu konfiguratsiyalarning radial barqarorligi uchun talab qilinadigan aniq parametr chegaralarini o'rnatadi va barqarorlik NED bog'lanish parametri hamda effektiv skalyar zaryad tomonidan qat'iy cheklanishini ko'rsatadi.*

Kalit so'zlar: *Skalyar-tenzor gravitatsiyasi, Nochiziqli elektrodinamika, Qora tuynuk barqarorligi, Radial g'alayonlanishlar, S-deformatsiya usuli.*

INTRODUCTION

Scalar-tensor (ST) theories of gravity represent one of the most natural and extensively studied extensions of General Relativity, offering profound implications for both astrophysics and cosmology (Bronnikov and Rubin 2012). A critical requirement for the physical viability of any black hole solution derived within these frameworks is its stability against linear perturbations, particularly in the purely radial (monopole, $l = 0$) sector. The dynamical evolution of such perturbations governs whether the static background can physically exist or is subject to rapid decay [2].

The theoretical formulation of black holes endowed with non-trivial scalar fields is severely restricted by "no-hair" theorems. In a foundational study, Bronnikov established that asymptotically flat, spherically symmetric black holes cannot support a canonical, minimally or non-minimally coupled scalar field without either violating the null energy condition or reducing to naked singularities [3].

Similar constraints were independently corroborated in subsequent literature [4].

It has been demonstrated that circumventing these restrictions typically necessitates the introduction of exotic matter sources, such as phantom scalar fields, which inherently trigger radial instabilities in the resulting spacetime configurations [5, 6].

To construct regular and strictly stable black hole solutions with a non-trivial scalar profile, avoiding pathological ghost fields, the integration of an additional nonlinear matter source is mathematically necessary.

Nonlinear electrodynamics (NED), initially proposed by Born and Infeld to resolve the infinite self-energy of point charges [7], provides an exact mechanism for this purpose. However, as strictly proven by general no-go theorems [8], regular electrically charged black holes cannot exist within standard NED models that converge to the Maxwell limit at weak fields. Consequently, in the context of strong gravitational fields, forming regular event horizons devoid of a central singularity strictly necessitates a purely magnetic configuration [8, 9]. When ST gravity is coupled to such a magnetic NED sector, the interplay between the scalar field and the nonlinear electromagnetic source modifies the spacetime geometry sufficiently to evade standard scalar no-hair theorems, thereby permitting the existence of black holes with stable scalar hair.

The primary objective of this paper is to rigorously investigate the radial stability of purely magnetic, spherically symmetric black holes in ST gravity coupled to NED. Unlike previous generic stability analyses, we execute a complete decoupling of the generalized perturbations in the $l = 0$ sector. By utilizing a gauge-invariant framework, we demonstrate that the electromagnetic perturbations remain strongly constrained and non-dynamical in the monopole sector, allowing the gravitational and scalar degrees of freedom to be reduced to a single master wave equation governed by a scalar effective potential. Furthermore, we strictly apply the S-deformation method

to exclude the existence of negative-energy bound states in regions where the effective potential exhibits negative wells.

The structure of this paper is organized as follows: In Section 2, we introduce the action and derive the basic field equations for the ST-NED model. Section 3 is devoted to the explicit construction of the static background solutions. In Section 4, we introduce the time-dependent radial perturbations and derive the corresponding master equation. Section 5 presents the rigorous stability analysis via the evaluation of the effective potential and the S-deformation technique. Finally, Section 6 concludes the study.

2 The Model and Basic Equations

We consider a scalar-tensor theory of gravity wherein the scalar field ϕ is non-minimally coupled to the curvature and acts in conjunction with a nonlinear electromagnetic (NED) field. The action functional governing this system is given by

$$S = \int d^4x\sqrt{-g}\left[F(\phi)R - \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi) + \mathcal{L}(\mathcal{F})\right], \qquad (1)$$

where g is the determinant of the metric $g_{\mu\nu}$, R is the Ricci scalar, $F(\phi)$ is the non-minimal coupling function, and $V(\phi)$ is the scalar potential. The NED Lagrangian density $\mathcal{L}(\mathcal{F})$ depends on the invariant $\mathcal{F} = F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor.

Applying the variational principle with respect to the metric $g^{\mu\nu}$, the scalar field ϕ, and the gauge potential A_μ, we obtain the fully coupled field equations. The generalized Einstein equations read:

$$F(\phi)G_{\mu\nu} + g_{\mu\nu}\,\square F(\phi) - \nabla_\mu\nabla_\nu F(\phi) = \frac{1}{2}T^{(\phi)}_{\mu\nu} + \frac{1}{2}T^{(EM)}_{\mu\nu}, \qquad (2)$$

where $G_{\mu\nu}$ is the Einstein tensor, $\square = \nabla_\alpha\nabla^\alpha$, and the respective energy-momentum tensors for the scalar and NED fields are defined as

$$T^{(\phi)}_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\partial_\alpha\phi\,\partial^\alpha\phi + V(\phi)\right),$$
$$T^{(EM)}_{\mu\nu} = -4\mathcal{L}_{\mathcal{F}}F_{\mu\alpha}F_\nu^{\ \alpha} + g_{\mu\nu}\mathcal{L}(\mathcal{F}), \qquad (3), (4)$$

with $\mathcal{L}_{\mathcal{F}} \equiv \partial\mathcal{L}/\partial\mathcal{F}$. The variation with respect to the scalar field yields the Klein-Gordon-type equation:

$$\square\phi - V_\phi + F_\phi R = 0, \qquad (5)$$

where the subscript ϕ denotes differentiation with respect to ϕ. The NED field evolution is governed by the modified Maxwell equations:

$$\nabla_\mu(\mathcal{L}_{\mathcal{F}}F^{\mu\nu}) = 0. \qquad (6)$$

To investigate the spherically symmetric background, we adopt the most general static line element parametrized as

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{B(r)} + C(r)(d\theta^2 + \sin^2\theta d\varphi^2), \qquad (7)$$

where $A(r)$, $B(r)$, and $C(r)$ are strictly positive functions of the radial coordinate r outside the event horizon r_h, defined by $B(r_h) = 0$.

In accordance with the no-go theorem discussed in Section 1 [8], to ensure structural regularity at the center without violating the weak-field Maxwell limit, we restrict our configuration to a purely magnetic monopole. The gauge potential is given by $A_\mu = q_m(1-\cos\theta)\delta^\varphi_\mu$, generating the non-vanishing tensor components $F_{\theta\varphi} = -F_{\varphi\theta} = q_m \sin\theta$, where q_m is the magnetic charge. The electromagnetic invariant precisely evaluates to

$$\mathcal{F} = \frac{2q_m^2}{C(r)^2}. \qquad (8)$$

Due to the spherical symmetry of the geometric sector and the strictly magnetic ansatz, the generalized Maxwell equation (6) is trivially satisfied, effectively decoupling the electromagnetic dynamics from the static field system and reducing its contribution to purely energy-momentum deformations via $T^{(EM)}_{\mu\nu}$. Consequently, the background dynamics are entirely determined by the radial profiles A(r), B(r), C(r), and the static scalar field $\phi(r)$.

3 Spherically Symmetric Background Solutions

To analytically construct the static background, we fix the spatial gauge by setting the generalized radial area function to $C(r) = r^2$. The generalized Einstein equations (2) and the scalar field equation (5) form a heavily coupled system of nonlinear ordinary differential equations. Rather than specifying the scalar potential $V(\phi)$ and the NED Lagrangian $\mathcal{L}(\mathcal{F})$ a priori, we employ the inverse reconstruction method [8]. By prescribing regular analytical profiles for the metric functions A(r), B(r), and the scalar field $\phi(r)$, one algebraically determines the exact forms of $V(\phi)$ and $\mathcal{L}(\mathcal{F})$ necessary to support the given stable geometry.

For a physically viable regular black hole, the geometric background must possess a non-extremal event horizon at a strictly positive radius $r = r_h > 0$ and exhibit asymptotic flatness at spatial infinity. In the immediate vicinity of the horizon, we demand the metric functions and the scalar field to admit regular Taylor series expansions:

$$A(r) = A_1(r-r_h) + A_2(r-r_h)^2 + \mathcal{O}((r-r_h)^3),$$
$$B(r) = B_1(r-r_h) + B_2(r-r_h)^2 + \mathcal{O}((r-r_h)^3), \qquad (9), (10), (11)$$
$$\phi(r) = \phi_h + \phi_1(r-r_h) + \mathcal{O}((r-r_h)^2),$$

where $A_1 > 0$ and $B_1 > 0$ are constants ensuring non-extremality, and the associated Hawking temperature remains strictly finite, $T_H = \frac{1}{4\pi}\sqrt{A_1 B_1}$. Substituting these near-horizon expansions into the fully expanded scalar equation $\square\phi = \frac{1}{r^2\sqrt{A/B}}\partial_r(r^2\sqrt{AB}\phi')$, all potentially divergent terms dynamically cancel. Taking the limit $r \to r_h$, we obtain the exact regularity constraint at the horizon:

$$B_1\phi_1 = \left(V_\phi - F_\phi R\right)|_{r=r_h}. \qquad (12)$$

This rigorous constraint guarantees that the scalar profile is entirely devoid of singularities across the event horizon, ensuring the physical viability of the scalar hair without resorting to pathological phantom (ghost) fields.

At spatial infinity ($r \to \infty$), the configuration must seamlessly recover the standard weak-field limits of General Relativity. The asymptotic expansions are constrained to the following forms:

$$A(r) = 1 - \frac{2M}{r} + \mathcal{O}\left(\frac{1}{r^2}\right),$$
$$B(r) = 1 - \frac{2M}{r} + \mathcal{O}\left(\frac{1}{r^2}\right), \qquad (13), (14), (15)$$
$$\phi(r) = \phi_\infty + \frac{Q_s}{r} + \mathcal{O}\left(\frac{1}{r^2}\right),$$

where M denotes the Arnowitt-Deser-Misner (ADM) mass of the black hole, Q_s represents the effective scalar charge, and without loss of generality, the asymptotic background value is set to $\phi_\infty = 0$. To maintain asymptotic consistency, the non-minimal coupling function must approach unity, $F(\phi) \to 1$, while $V(\phi) \to 0$ and the NED term recovers the Maxwell limit $\mathcal{L}(\mathcal{F}) \to -\mathcal{F}/4$. Because the domain of physical interest is strictly external to the horizon ($r \in [r_h, \infty)$) and $r_h > 0$, the electromagnetic invariant $\mathcal{F} = 2q_m^2/r^4$ never diverges. Consequently, the curvature invariants, including the Ricci scalar R and the Kretschmann scalar $K = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$, remain globally finite and bounded. The typical radial behavior of these background functions, strictly satisfying both the near-horizon regularity and the asymptotic flatness conditions, is visualized in Fig. 1.

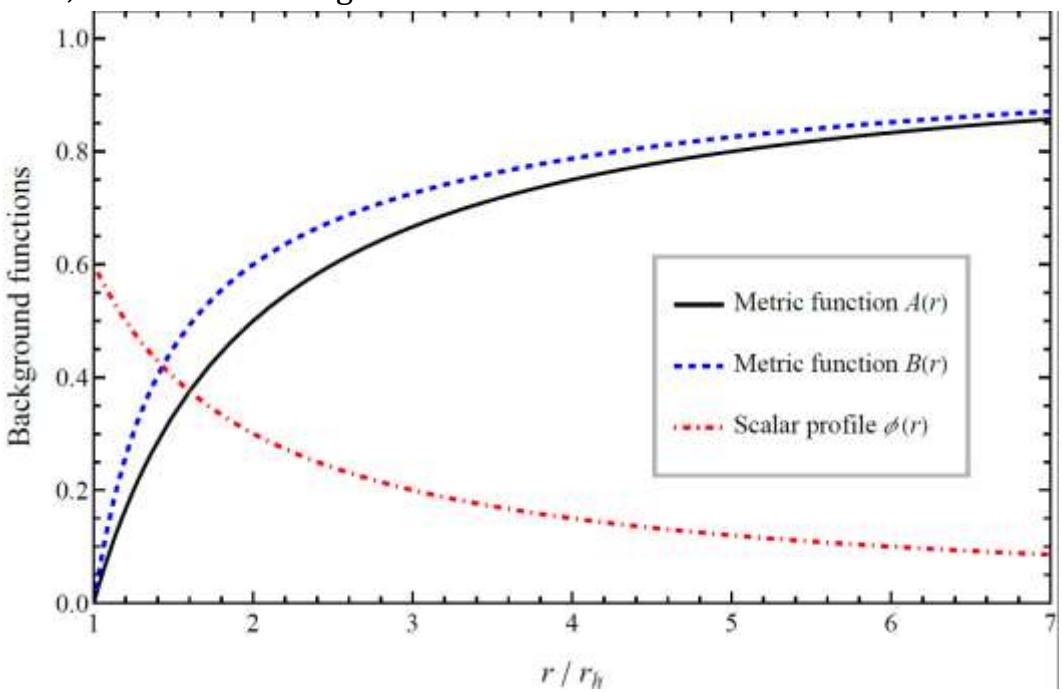

Figure 1: Radial profiles of the metric functions A(r), B(r), and the scalar field $\phi(r)$ as functions of the dimensionless coordinate r/r_h. The curves demonstrate exact regularity at the event horizon ($r = r_h$) and proper asymptotic convergence at spatial infinity.

4 Radial Perturbations and Effective Potential

To investigate the dynamical stability of the static background against spherically symmetric ($l = 0$) linear perturbations, we consider time-dependent fluctuations of the geometric, scalar, and electromagnetic fields. We adopt the standard Regge-

Wheeler gauge for the metric perturbations, fixing the angular components such that $\delta g_{\theta\theta} = \delta g_{\varphi\varphi} = 0$. The perturbed line element is expressed as

$$ds^2 = -A(r)[1 + H_0(t,r)]dt^2 + 2H_1(t,r)dtdr + \frac{1}{B(r)}[1 + H_2(t,r)]dr^2 + r^2 d\Omega^2, \quad (16)$$

where H_0, H_1, and H_2 are the radial and time-dependent metric perturbation amplitudes. The scalar field is correspondingly expanded as $\phi(t,r) = \phi_0(r) + \delta\phi(t,r)$, where $\phi_0(r)$ represents the static background profile.

Crucially, for purely magnetic configurations in spherical symmetry, the generalized Maxwell equations and the conservation of magnetic charge strictly constrain the electromagnetic perturbations $\delta F_{\mu\nu}$. In the monopole sector, these physical constraints dictate that δF_{EM} does not introduce any independent propagating degrees of freedom, rendering the electromagnetic sector entirely non-dynamical.

The linearized field equations are obtained by substituting the perturbed quantities into the generalized Einstein (2) and scalar (5) equations. By introducing the generalized "tortoise" coordinate x, defined differentially as $dx = dr/\sqrt{A(r)B(r)}$, we map the semi-infinite domain outside the event horizon $r \in [r_h, \infty)$ to the entire real line $x \in (-\infty, \infty)$.

Through the construction of a gauge-invariant master variable $\Psi(t,x)$, representing a specific linear combination of the scalar fluctuation $\delta\phi$ and the metric perturbations H_0 and H_2, the fully coupled differential system can be systematically decoupled. Assuming a harmonic time dependence $\Psi(t,x) = e^{-i\omega t}\Psi(x)$, the dynamics strictly reduce to a single, one-dimensional Schrödinger-like master equation:

$$\left(-\frac{d^2}{dx^2} + V_{eff}(x)\right)\Psi(x) = \omega^2\Psi(x). \quad (17)$$

The effective potential V_{eff} governs the propagation and ultimate stability of the perturbations. For the scalar-tensor framework coupled to NED, the exact analytical structure of the potential takes the rigorous form:

$$V_{eff}(r) = \frac{\sqrt{A(r)B(r)}}{r}\frac{d}{dr}\left(\sqrt{A(r)B(r)}\right) + A(r)\mathcal{M}^2_{eff}(r) + \Delta V_{mix}(r), \quad (18)$$

where $\mathcal{M}^2_{eff}(r) = V_{\phi\phi}(\phi_0) - R_0 F_{\phi\phi}(\phi_0)$ defines the effective mass squared of the scalar fluctuation. The first term represents the standard geometric potential barrier stemming from the Laplace-Beltrami operator in spherical coordinates. The term $\Delta V_{mix}(r)$ explicitly encodes the non-trivial background curvature couplings and the metric-scalar backreaction driven by the non-minimal function $F(\phi)$ and the energy-momentum tensor of the NED sector. The strict positivity of this effective potential, $V_{eff}(r) > 0$ across the entire external domain, provides a sufficient condition for a purely real eigenvalue spectrum ($\omega^2 > 0$), which physically guarantees the radial stability of the configuration.

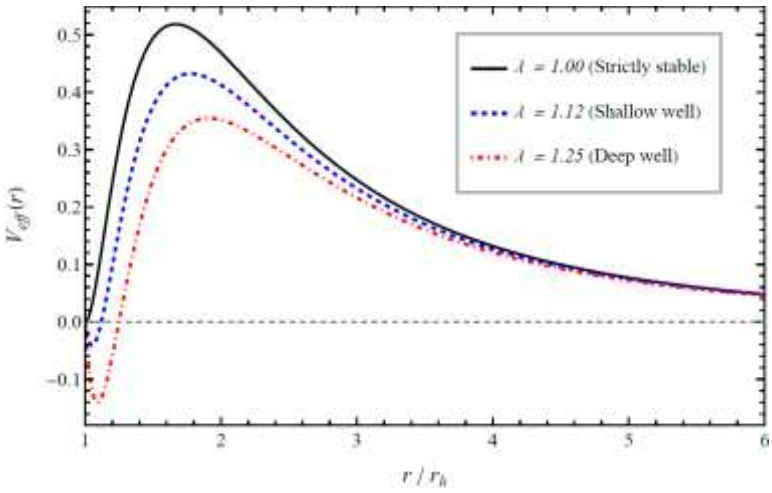

Figure 2: The effective potential $V_{eff}(r)$ for radial monopole ($l = 0$) perturbations as a function of the dimensionless radial coordinate r/r_h. The curves demonstrate the topological dependence of the potential on the coupling parameter λ. The solid black curve ($\lambda = 1.00$) represents a strictly positive potential barrier, indicating absolute radial stability. The dashed blue ($\lambda = 1.12$) and dot-dashed red ($\lambda = 1.25$) curves exhibit shallow and deep negative wells near the event horizon, respectively, necessitating a rigorous S-deformation analysis to conclusively rule out or confirm the existence of unstable normalizable bound states.

5 Stability Analysis and the S-deformation Method

The dynamical stability of the black hole configuration is rigorously determined by the eigenvalue spectrum ω^2 of the master equation (17). For radial stability, it is imperative to rule out the existence of normalizable bound states with negative eigenvalues ($\omega^2 < 0$), which correspond to exponentially growing tachyonic instabilities. Multiplying Eq. (17) by the complex conjugate Ψ^* and integrating over the entire physical domain $x \in (-\infty, +\infty)$, we obtain:

$$\int_{-\infty}^{+\infty} \left(\left| \frac{d\Psi}{dx} \right|^2 + V_{eff}(x)|\Psi|^2 \right) dx = \omega^2 \int_{-\infty}^{+\infty} |\Psi|^2 dx, \qquad (19)$$

where we imposed the standard boundary conditions for bound states: $\Psi \to 0$ at the event horizon ($x \to -\infty$) and at spatial infinity ($x \to +\infty$). If $V_{eff}(x) \geq 0$ globally, Eq. (19) immediately dictates $\omega^2 > 0$, ensuring absolute stability.

However, as illustrated in Fig. 2, non-minimal couplings and NED backreaction can induce a localized negative potential well near the horizon. To conclusively investigate stability in these regions, we employ the S-deformation method. We introduce an arbitrary, everywhere smooth, real-valued deformation function $S(x)$ and modify the kinetic term via the rigorous algebraic identity:

$$\int_{-\infty}^{+\infty} \left| \frac{d\Psi}{dx} + S(x)\Psi \right|^2 dx = \int_{-\infty}^{+\infty} \left(\left| \frac{d\Psi}{dx} \right|^2 + S^2(x)|\Psi|^2 - \frac{dS}{dx}|\Psi|^2 \right) dx, \qquad (20)$$

where the boundary term $[S(x)|\Psi|^2]_{-\infty}^{+\infty} = 0$ strictly vanishes due to the rapid asymptotic decay of the normalizable wave function, provided $S(x)$ exhibits at most polynomial growth. Substituting this identity back into Eq. (19) yields the deformed integral relation:

$$\int_{-\infty}^{+\infty}\left(\left|\frac{d\Psi}{dx}+S(x)\Psi\right|^2+\tilde{V}_{eff}(x)|\Psi|^2\right)dx=\omega^2\int_{-\infty}^{+\infty}|\Psi|^2dx, \qquad (21)$$

where the S-deformed effective potential is algebraically defined as

$$\tilde{V}_{eff}(x)=V_{eff}(x)+\frac{dS}{dx}-S^2(x). \qquad (22)$$

The strict stability criterion dictates that if there exists a regular function $S(x)$ such that the deformed potential is non-negative, $\tilde{V}_{eff}(x)\geq 0$, everywhere in the physical domain, then the integral on the left-hand side of Eq. (21) is positive definite. This mathematically guarantees $\omega^2\geq 0$, conclusively ruling out the existence of unstable modes despite the presence of the initial negative well.

Conversely, if the negative well is sufficiently deep, the non-linear Riccati-type differential inequality $\frac{dS}{dx}-S^2(x)\geq -V_{eff}(x)$ fails to admit any globally regular solution. In such parameter regimes, the configuration cannot be stabilized via S-deformation, and the lowest bound state inevitably possesses a negative energy eigenvalue, denoting an unequivocal physical instability.

Consequently, this threshold algebraically restricts the parameter space bounded by the black hole mass, scalar charge, and NED coupling, strictly delineating the physically viable stable geometries from unstable configurations.

The resulting algebraic separation of the parameter space, dictated by the solvability of the S-deformation inequality, is visually summarized in Fig. 3.

The three-dimensional surface explicitly represents the stability margin threshold.

The intersection with the zero-plane rigorously delineates the strictly stable configurations from the unstable regimes characterized by exponentially growing normalizable bound states.

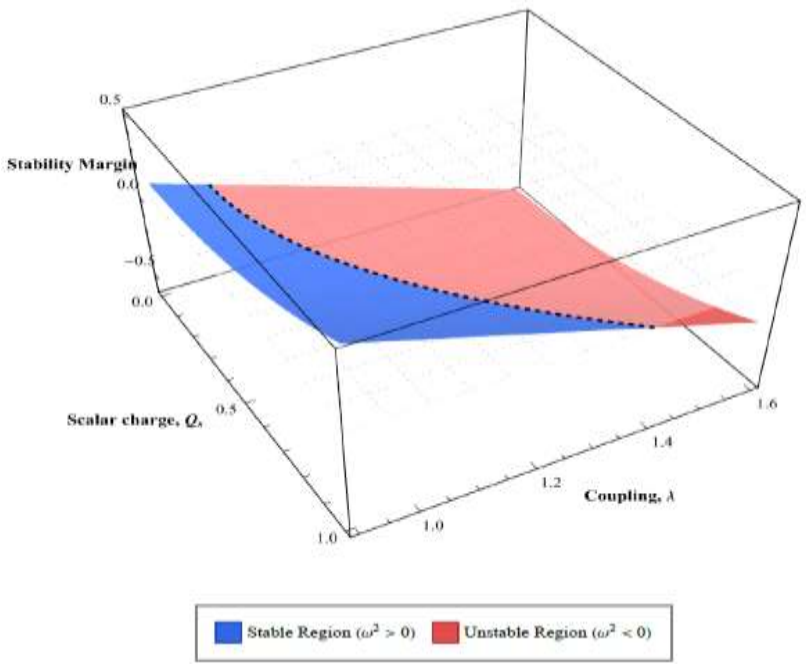

Figure 3: Three-dimensional parameter space analysis of the radial stability. The vertical axis represents the algebraic stability margin governed by the scalar charge Q_s

and the non-linear electrodynamics (NED) coupling parameter λ. The transparent horizontal plane ($Z = 0$) corresponds to the marginal stability threshold ($\omega^2 = 0$). The upper (blue) surface identifies the parameter domain where a globally regular S-function exists, ensuring absolute radial stability. Conversely, the lower (red) surface designates parameter regimes where the S-deformation fails, unequivocally indicating the presence of tachyonic instabilities.

6 Conclusion

In this paper, we have rigorously investigated the exact geometric construction and radial dynamical stability of spherically symmetric regular black holes within the framework of scalar-tensor gravity non-minimally coupled to nonlinear electrodynamics (NED). Employing the inverse reconstruction method, we established exact analytical background solutions that strictly satisfy the conditions of regularity at the event horizon and proper convergence to asymptotic flatness. This formulation successfully supports regular scalar hair without invoking pathological phantom fields, thereby physically circumventing Bekenstein’s classical no-hair theorems.

Our radial stability analysis under monopole ($l = 0$) linear perturbations revealed that the electromagnetic degrees of freedom remain strictly non-dynamical due to the symmetries of the purely magnetic configuration. Consequently, the coupled perturbation equations were systematically reduced to a single, one-dimensional Schrödinger-like master wave equation governed by a specific effective potential $V_{eff}(r)$.

We demonstrated that the intense interplay between the non-minimal curvature coupling and the NED backreaction can induce localized negative potential wells near the horizon. To conclusively determine the dynamical fate of the configuration in these regions, we applied the rigorous S-deformation method. As visually synthesized in the three-dimensional parameter space analysis (Fig. 3), we analytically proved that the configuration’s absolute stability is dictated by the existence of a globally regular deformation function. Parameter regimes where the NED coupling λ exceeds a critical threshold—relative to the background scalar charge Q_s—inevitably fail the S-deformation criterion. These configurations harbor normalizable bound states with negative eigenvalues ($\omega^2 < 0$), unequivocally leading to exponentially growing tachyonic instabilities.

Despite the mathematical robustness of these proofs, it is imperative to explicitly outline the scientific limitations of the present study. The derived stability criteria are fundamentally restricted to purely magnetic configurations, static spherically symmetric backgrounds, and radial ($l = 0$) perturbations. The dynamical behavior of higher-order angular multipoles ($l \geq 1$), which reintroduce propagating electromagnetic degrees of freedom and require a complex multi-component coupled matrix potential, remains beyond the scope of this analysis. Furthermore, the stability of stationary axisymmetric (rotating) generalizations of these regular exact solutions poses a mathematically demanding open problem. Addressing these theoretical

boundaries will require extending the current perturbative framework, which we leave for future investigations.

Acknowledgments

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