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TO'RTINCHI TARTIBLI BUZILADIGAN ODDIY DIFFERENSIAL TENGLAMA UCHUN SPEKTRAL MASALA

Azizov Muzaffar Sulaymonovich
FarDU, matematik analiz va differensial tenglamalar kafedrasi katta o'qituvchisi;
Xusanova Xusnida Qahramonovna
FarDU, matematika yo'nalishi 3-bosqich talabasi;
Kimsanboyeva Shohsanam Mirzohid qizi
FarDU, amaliy matematika yo'nalishi 3-bosqich talabasi
kimsanboyevashohsanam@gmail.com

Annotatsiya. *Ushbu maqolada kesmaning buziladigan to'rtinchi tartibli oddiy differensial tenglama uchun spektal masala bayon qilingan va bu masala yechimi Grin funksiyalar usuli yordamida xos sonlar va ularga mos xos funksiyalarga ega ekanligi isbotlangan.*

Kalit so'zlar: *buziladigan tenglama, chegaraviy masala, Grin funksiya, Gilbert teoremasi, xos son, xos funksiya.*

Kirish

Yuqori tartibli differensial tenglamalar ko'plab fizikaviy va muhandislik jarayonlarini modellashtirishda muhim rol o'ynaydi. Jumladan, elastiklik nazariyasi, issiqlik uzatish, kvant mexanikasi, to'lqin tarqalishi va boshqa sohalarda bunday tenglamalarning tabiiy ravishda paydo bo'lishi kuzatiladi. Ayniqsa, og'irlik funksiyalari bilan differensial operatorlar o'zgaruvchan muhitda sodir bo'ladigan jarayonlarni aniqlik bilan modellashtirish imkonini beradi. Bunda spektral masalalar - ya'ni differensial operatorlarga mos xos qiymatlar va xos funksiyalarni topish masalasi — markaziy o'rinni egallaydi. Bunday spektral masalalarni o'rganishning ilk asosiy yondashuvlari 20-asrning o'rtalarida shakllanib, shundan buyon turli murakkablashtirilgan va umumlashtirilgan holatlarda keng tadqiq qilinmoqda. Jumladan, V.S. Vladimirov [1], A.A. Samarskii [2], S.L. Sobolev [3], V.A. Il'in [4] va boshqa olimlar klassik va noan'anaviy shartlar ostida differensial operatorlarning spektral xossalarini chuqur o'rganganlar.

So'nggi yillarda bu sohaga bag'ishlangan o'zbek olimlarining ham sezilarli hissasi bo'lmoqda. Xususan, Azizov M.S. va Urinov A.K. tomonidan olib borilgan tadqiqotlar [5], [6] yuqori juft tartibli, og'irlik funksiyalari ishtirokidagi differensial tenglamalar uchun boshlang'ich-chegaraviy masalalar spektral nazariyasini rivojlantirishga bag'ishlangan. Ular Bessel operatori ishtirokidagi tenglamalarni o'rganib, ularning xos qiymatlari va xos funksiyalarining mavjudligini, yagonaligini va turg'unligini o'zgaruvchilarni ajratish usuli yordamida aniqlashgan.

Biz ko'rib chiqayotgan quyidagi spektral masala

$$L[y]=\left(x^{\alpha}y''(x)\right)''=\lambda y(x),\ 0<\alpha<1 \qquad (1)$$

$$y(0)=0,\quad y'(0)=0,\quad \left(x^{\alpha}y''(x)\right)\Big|_{x=1}=0,\quad \left(x^{\alpha}y''(x)\right)'\Big|_{x=1}=0 \qquad (2)$$

ham o'zida yuqori tartibli differensial operatorni va nostandart og'irlik funksiyasini mujassam etgani uchun yuqoridagi ishlarga tabiiy tarzda uzviy bog'liqdir.

Ushbu spektral masala yechimlarini o'rganish uchun yuqoridagi klassik va zamonaviy natijalar tayanch sifatida xizmat qiladi.

{(1)-(2)} spektrial masalani qaraylik.

Dastlab, {(1),(2)} masalaning o'z-o'ziga qo'shma ekanligini isbotlaylik. Agar $y(x),w(x)\in C^{3}[0,1]\cap C^{4}(0,1)$ va $y^{4}(x),\ w^{4}(x)\in L[0,1]$ bo'lsa, u holda quyidagi bo'laklab integrallangan ifoda o'rinli.

$$\int_0^1\left(x^{\alpha}y''(x)\right)''w(x)dx=\int_0^1 y(x)\left(x^{\alpha}w''(x)\right)''dx+\left(x^{\alpha}y''(x)\right)'\Big|_{x=1}w(1)-$$

$$-\left(x^{\alpha}y''(x)\right)''\Big|_{x=0}w(0)-\left[\left(x^{\alpha}y''(x)\right)\Big|_{x=1}w'(1)-\left(x^{\alpha}y''(x)\right)\Big|_{x=0}w'(0)\right]+$$

$$+y'(1)\left(x^{\alpha}w''(x)\right)\Big|_{x=1}-y'(0)\left(x^{\alpha}w''(x)\right)\Big|_{x=0}-$$

$$-\left[y(1)\left(x^{\alpha}w''(x)\right)'\Big|_{x=1}-y(0)\left(x^{\alpha}w''(x)\right)'\Big|_{x=0}\right].$$

Bundan kelib chiqadiki, $y(x)$ va $w(x)$ (2) shartlarni qanoatlantirsa, $\int_0^1\left(x^{\alpha}y''(x)\right)''w(x)dx=\int_0^1 y(x)\left(x^{\alpha}w''(x)\right)''dx$ tenglik kelib chiqadi, bu esa {(1),(2)} masalaning o'z-o'ziga qo'shma masala ekanligini ko'rsatadi.

(1) tenglamani $y(x)$ ga ko'paytirib, $[0,1]$ kesmada integrallaymiz.

$$\int_0^1\left(x^{\alpha}y''(x)\right)''y(x)dx=\lambda\int_0^1 y^{2}(x)dx.$$

Oxirgi tenglikning chap tomonidagi integralda bo'laklab integrallash qoidasini ikki marta qo'llasak, ushbu tenglik hosil bo'ladi.

$$\lambda\int_0^1 y^{2}(x)dx=\left[y(x)\left(x^{\alpha}y''(x)\right)'-y'(x)x^{\alpha}y''(x)\right]\Big|_0^1+\int_0^1\left(\sqrt{x^{\alpha}}\,y''(x)\right)^{2}dx.$$

Oxirgi tenglikda (2) shartlarni e'tiborga olsak, u ushbu

$$\lambda\int_0^1 y^{2}(x)dx=\int_0^1\left(\sqrt{x^{\alpha}}\,y''(x)\right)^{2}dx.$$

ushbu ko'rinishni oladi. Bu tenglikdan $\lambda\geq 0$ ekanligi kelib chiqadi.

$\lambda = 0$ bo‘lgan holni alohida qaraylik. Bu holda (1) tenglama $L[y] = \left(x^{\alpha}\, y''(x)\right)'' = 0$ ko‘rinishni olib, uning umumiy yechimi

$$y(x) = C_1 \frac{x^{3-\alpha}}{(3-\alpha)(2-\alpha)} + C_2 \frac{x^{2-\alpha}}{(2-\alpha)(1-\alpha)} + C_3\, x + C_4$$

ko‘rinishda aniqlanadi. Bu yerda $C_j,\, j = \overline{1,4}$ - ixtiyoriy o‘zgarmaslar.

Bu yechimni (2) shartlarga bo‘ysundirib, $C_j = 0,\ j = \overline{1,4}$ tengliklarni hosil qilamiz. Bundan kelib chiqadiki, $\{(1),(2)\}$ masala faqatgina $\lambda > 0$ bo‘lgandagina notrivial yechimga ega bo‘ladi.

Endi $\{(1),(2)\}$ masalaning xos sonlari va xos funksiyalari mavjudligini ko‘rsatamiz. Shu maqsadda $\{(1),(2)\}$ masalaning Grin funksiyasini quramiz. Bu masalaning Grin funksiyasi quyidagi ko‘rinishda aniqlanadi:

$$G(x,s) = \begin{cases} -\dfrac{s^{3-\alpha}x^{3-\alpha}}{3-\alpha} + \dfrac{s^{2-\alpha}x^{3-\alpha}}{2-\alpha} - \dfrac{x^{3-\alpha}}{(3-\alpha)(2-\alpha)} - \\ -\dfrac{s^{2-\alpha}x^{2-\alpha}}{1-\alpha} + \dfrac{s\,x^{2-\alpha}}{(2-\alpha)(1-\alpha)}, & 0 \le x < s, \\ -\dfrac{x^{3-\alpha}s^{3-\alpha}}{3-\alpha} + \dfrac{x^{2-\alpha}s^{3-\alpha}}{2-\alpha} - \dfrac{s^{3-\alpha}}{(3-\alpha)(2-\alpha)} - \\ -\dfrac{x^{2-\alpha}s^{2-\alpha}}{1-\alpha} + \dfrac{x\,s^{2-\alpha}}{(2-\alpha)(1-\alpha)}, & s < x \le 1. \end{cases} \qquad (3)$$

Grin funksiya $L[y] \equiv \left(x^{\alpha}\, y''(x)\right)'' = 0$ tenglamani va (2) chegaraviy shartlarni qanoatlantirib, x bo‘yicha ikkinchi tartibgacha hosilasi $\forall x = s \in (0,1)$ da uzluksiz, uchinchi tartibli hosilasi esa

$$\left(\partial^3/\partial x^3\right)G(s,s+0) - \left(\partial^3/\partial x^3\right)G(s,s-0) = 1$$

ko‘rinishda sakrashga ega. U holda, $L[y] = f(x)$ tenglamani va (2) chegaraviy shartlarni qanoatlantiradigan $y(x)$ funksiya

$$y(x) = \int_0^1 G(x,s)\, f(s)\, ds \qquad (4)$$

ko‘rinishda aniqlanadi, bu yerda $f(x)$ berilgan uzluksiz funksiya. Haqiqatan ham,

(4) funksiya $L[y] = f(x)$ tenglamani qanoatlantirishini ko‘rsatamiz. Buning uchun avval (4) funksiyani

$$y(x)=\int_{0}^{x-0}G(x,s)f(s)ds+\int_{x+0}^{1}G(x,s)f(s)ds$$

ko‘rinishda yozib olamiz va undan 4-tartibli hosilalarigacha olib, ketma-ket hisoblaymiz. Bunda $G(x,s)$ funksiyaning xossalarini e’tiborga olib,

$$y'(x)=\int_{0}^{x-0}G'_x(x,s)f(s)ds+\int_{x+0}^{1}G'_x(x,s)f(s)ds+$$
$$+G(x,x-0)f(x-0)-G(x,x+0)f(x+0)=$$
$$=\int_{0}^{x-0}G'_x(x,s)f(s)ds+\int_{x+0}^{1}G'_x(x,s)f(s)ds,$$

$$y''(x)=\int_{0}^{x-0}G''_x(x,s)f(s)ds+\int_{x+0}^{1}G''_x(x,s)f(s)ds+$$
$$+G'_x(x,x-0)f(x-0)-G'_x(x,x+0)f(x+0)=$$
$$=\int_{0}^{x-0}G''_x(x,s)f(s)ds+\int_{x+0}^{1}G''_x(x,s)f(s)ds,$$

$$y'''(x)=\int_{0}^{x-0}G'''_x(x,s)f(s)ds+\int_{x+0}^{1}G'''_x(x,s)f(s)ds+$$
$$+G''_x(x,x-0)f(x-0)-G''_x(x,x+0)f(x+0)=$$
$$=\int_{0}^{x-0}G'''_x(x,s)f(s)ds+\int_{x+0}^{1}G'''_x(x,s)f(s)ds,$$

$$y^{IV}(x)=\int_{0}^{x-0}G^{IV}_x(x,s)f(s)ds+\int_{x+0}^{1}G^{IV}_x(x,s)f(s)ds+$$
$$+G'''_x(x,x-0)f(x-0)-G'''_x(x,x+0)f(x+0)=$$
$$=\int_{0}^{x-0}G^{(IV)}_x(x,s)f(s)ds+\int_{x+0}^{1}G^{(IV)}_x(x,s)f(s)ds+f(x).$$

tengliklarga ega bo‘lamiz. Demak, $y(x)$ funksiya $Ly(x)=f(x)$ tenglamani qanoatlantiradi. (4) funksiyani (2) shartlarni qanoatlantirishini, $G(x,s)$ funksiyaning xossalaridan bevosita kelib chiqadi.

Yuqoridagilardan ko‘rinib turibdiki, $\{(1),(2)\}$ masala

$$y(x)=\lambda\int_{0}^{1}G(x,s)y(s)ds \qquad (5)$$

integral tenglamaga ekvivalent ekan.

$G(x,s)$ – simmetrik, uzluksiz va musbat yadro bo'lganligi uchun, simmetrik yadroli integral tenglamalar nazariyasiga asosan (5) integral tenglama sanoqli sondagi

$0<\lambda_1<\lambda_2<\lambda_3<\ldots$

xos sonlarga va ularga mos ortonormal

$$v_1(x), v_2(x), v_3(x), \ldots \tag{6}$$

xos funksiyalarga ega bo'ladi. Shuningdek, har qanday $g(x)\in L_2[0,1]$ funksiya bu xos funksiyalar bo'yicha o'rtacha yaqinlashuvchi qatorga yoyiladi.

Lemma. *Agar* $g(x)$ *funksiya* $g(x), x^{\alpha}g''(x)\in C[0,1]\cap C^1[0,1]$, $\left(s^{\alpha}g''(s)\right)''\in L_2(0,1)$ *sinfga tegishli bo'lsa, va (2) shartlarni bajarsa, u holda* $g(x)$ *funksiya (6) xos funksiyalar bo'yicha [0,1] intervalda absolyut va tekis yaqinlashuvchi*

$$g(x)=\sum_{n=1}^{+\infty} g_n v_n(x) \tag{7}$$

Furye qatoriga yoyiladi, bu yerda $g_n=\int_0^1 g(x)v_n(x)dx$, $n\in N$, *Furye koeffitsiyenti.*

Isbot. $g(x)$ va $G(x,s)$ funsiyalarga qo'yilgan shartlar xossalariga asosan quyidagi tenglik o'rinli bo'ladi:

$$g(x)=\int_0^1 G(x,s)\left(s^{\alpha}g''(s)\right)'' ds. \tag{8}$$

(8) tenglikni to'rt martta bo'laklab integrallab $g(x)$ va $G(x,s)$ funksiyalarning xossalarini e'tiborga olib, quyidagiga ega bo'lamiz:

$$g(x)=\int_0^{x-0} G(x,s)\left(s^{\alpha}g''(s)\right)'' ds+\int_{x+0}^{1} G(x,s)\left(s^{\alpha}g''(s)\right)'' ds=$$

$$=G(x,s)\left(s^{\alpha}g''(s)\right)'\Big|_{s=0}^{s=x-0}+G(x,s)\left(s^{\alpha}g''(s)\right)'\Big|_{s=x+0}^{s=1}-$$

$$-G'_s(x,s)\left(s^{\alpha}g''(s)\right)\Big|_{s=0}^{s=x-0}-G'_s(x,s)\left(s^{\alpha}g''(s)\right)\Big|_{s=x+0}^{s=1}+$$

$$+\left(s^{\alpha}G'_s(x,s)\right)g'(s)\Big|_{s=0}^{s=x-0}+\left(s^{\alpha}G'_s(x,s)\right)g'(s)\Big|_{s=x+0}^{s=1}-$$

$$-\left(s^{\alpha}G'_s(x,s)\right)' g(s)\Big|_{s=0}^{s=x-0}-\left(s^{\alpha}G'_s(x,s)\right)' g(s)\Big|_{s=x+0}^{s=1}+$$

$$+\int_0^1\left(s^{\alpha}G''_s(x,s)\right)'' g(s)ds=g(x).$$

Bundan kelib chiqadiki, $g(x)$ - $G(x,s)$ yadro orqali ifodalanuvchi funksiya. U holda, $g(x)$ funksiya (6) xos funksiyalar bo‘yicha absolyut va tekis yaqinlashuvchi (7) qatorga yoyiladi [7]. Lemma isbotlandi.

ADABIYOTLAR

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