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YIG'INDILARNI HISOBLASHDA GEOMETRIK USULDAN FOYDALANISH

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Annotatsiya

Ushbu maqolada asosi arifmetik progressiyani, darajasi esa istalgan natural sondan iborat bo'lgan sonli qatorlarning yig'indisini topishning bir necha usullari keltirilgan. Oson ko'rinishdagi yig'indilardan boshlab, murakkab ko'rinishdagi yig'indilarni hisoblash formulalari keltirib chiqarishda asosan geometrik usulga ko'proq e'tibor qaratilgan. Olingan natijalar umumiy holda ham keltirilgan.

Kalit so`zlar

Sonli qatorlar, Nuyuton binomi, Binomial koeffitsent, Arifmetik progressiya, qisqa ko'paytirish formulasi, to'g'ri to'rtburchak, kvadrat.

ПРИМЕНЕНИЕ ГЕОМЕТРИЧЕСКОГО МЕТОДА ПРИ ВЫЧИСЛЕНИИ СУММ

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В данной статье представлено несколько способов нахождения суммы числовых рядов, у которого степень произвольное натуральное число, а на основании, которого лежит арифметическая прогрессия. Начиная с простых вид сумм, при выводе формул для вычисления сложных вид сумм больше внимания уделено преимущественно геометрическому методу. Полученные результаты также представлены в общем виде.

Ключевые слова

числовые ряды, бином Ньютона, биномиальный коэффициент, арифметическая прогрессия, формула краткого умножения, прямоугольник, квадрат.

APPLICATION OF THE GEOMETRIC METHOD TO CALCULATE SUMS

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Annotation

This article presents several ways of finding of the sum of numerical series, in which the degree is an arbitrary natural number, and on the basis of which the arithmetic progression lies. Starting with simple types of sum, when deriving formulas for calculating complex types of sum, more attention is paid mainly to the geometric method.The obtained results are also presented in general terms
Keywords

number series, Newton binomial, binomial coefficient, arithmetic progression, formula of short multiplication, rectangle, square.

Ko‘plab masalalarni hal etish jarayonida ba’zi murakkab ko’rinishdagi yig‘indilarni hisoblashga to‘g‘ri keladi. Bunday vaqtda ko‘pincha tayyor formulalardan foydalanamiz. Lekin bu formulalarni qayerdan kelib chiqqanligiga qiziqmaydigan matematik bo‘lmasa kerak. Shu bois bunday yig‘indilar asrlar davomida muntazam o‘rganilib kelingan va o‘zining bir qancha isbotini topgan.

Shu yig‘indilardan bir guruhini isbotlarini misollar yordamida ya’ni hususiy holda, hamda umumiy holda ham keltirib o‘tamiz.

1-misol. $A_n^1=\sum_{k=0}^{n-1}(a+kd)=a+(a+d)+(a+2d)+...+(a+(n-1)d)$ (1)

ni hisoblang.

Bu yerda A_n^1 birinchi hadi a ga, ayirmasi d ga teng bo‘lgan arifmetik progressiyaning n ta hadi yig‘indisidan iborat. Shuning uchun

$$A_n^1=\frac{2a+(n-1)d}{2}n$$

ekanligi ko‘pchilikka ma’lum va bu formuladan doimiy foydalanib kelganmiz. Lekin bu formulaning ba’zi isbotlarini hamma joyda ham uchratavermaymiz. Bu misolning yechimini keyinroqga qoldirgan holda misolimizni ozroq qiyinlashtirsak.

2-misol. $A_n^2=\sum_{k=0}^{n-1}(a+kd)^2=a^2+(a+d)^2+(a+2d)^2+...+(a+(n-1)d)^2$ ni hisoblang.

Bu yig‘indining hadlari 1-misoldagi arifmetik progressiyaning mos hadlarining kvadratlaridan tuzilgan. Endi bu misolning yechimi anchagina murakkab deyishimiz mumkin. Agar

$$A_n^3=\sum_{k=0}^{n-1}(a+kd)^3=a^3+(a+d)^3+(a+2d)^3+...+(a+(n-1)d)^3$$

ko‘rinishdagi yig‘indini hisoblashni maqsad qilsak, bu yanada murakkablik tug‘diradi.

Biz endi istalgan A_n^m $(m\in N)$ larni topish usullarini keltiramiz.

3-misol. $A_n^m=\sum_{k=0}^{n-1}(a+kd)^m=a^m+(a+d)^m+(a+2d)^m+...+(a+(n-1)d)^m$

ni hisoblang.

1-usul: Bu usulda Nuyuton binomi formulasidan foydalaniladi.

$(a+b)^m=\sum_{k=0}^{m}C_m^k a^{m-k}b^k$, bu yerda $C_m^k=\frac{m(m-1)\dots(m-k+1)}{k!}$ binomial koeffitsient. Bu formuladan foydalanib quyidagilarni yozib olamiz.

$$(a+d)^{m+1}-a^{m+1}=C_{m+1}^1a^md+C_{m+1}^2a^{m-1}d^2+\dots+d^{m+1}$$

$$(a+2d)^{m+1}-(a+d)^{m+1}=C_{m+1}^1(a+d)^md+C_{m+1}^2(a+d)^{m-1}d^2+\dots+d^{m+1}$$

…………………………..

$$(a+nd)^{m+1}-(a+(n-1)d)^{m+1}=C_{m+1}^1(a+(n-1)d)^md+C_{m+1}^2(a+(n-1)d)^{m-1}d^2+\dots+d^{m+1}$$

Hosil bo‘lgan ifodalarni chap va o‘ng qismlarini qo‘shib, quyidagilarni olamiz [2].

$$(a+nd)^{m+1}-a^{m+1}=C_{m+1}^1A_n^md+C_{m+1}^2A_n^{m-1}d^2+C_{m+1}^3A_n^{m-2}d^3\dots+nd^{m+1} \qquad (2)$$

Endi (2) formuladan A_n^0 ni hisoblaymiz. Binomial koefsiyentdan agar $k>m+1$ bo‘lsa, $C_{m+1}^k=0$ ekanligi ma’lum. Shuning uchun, $m=0$ da (2) ifoda $(a+nd)^1-a^1=C_1^1A_n^0d$ tenglikka kelib qoladi. Bundan $A_n^0=n$ ekanligi kelib chiqadi. Agar (2) ifodaning $m=1$ dagi qiymatini hisoblasak

$$(a+nd)^2-a^2=C_2^1A_n^1d+C_2^2A_n^0d^2$$

$$A_n^1=\frac{2a+(n-1)d}{2}n$$

ekanligini topamiz. Bu natija birinchi misolimizdagi (1) arifmetik progressiyaning hadlari yig‘indisi formulasini isbotlaydi. Shu bilan 1-misol o‘z yechimini topdi desak bo‘ladi.

2-misolimizni ham shu tariqa yechishimiz mumkin. Buning uchun (2) ifodadagi m ning o‘rniga 2 ni qo‘yamiz.

$$(a+nd)^3-a^3=C_3^1A_n^2d+C_3^2A_n^1d^2+C_3^3A_n^0d^3$$

$$(a+nd)^3-a^3=3A_n^2d+2A_n^1d^2+A_n^0d^3 \qquad (3)$$

(3) ifodaga A_n^0 va A_n^1 ning qiymatlarini olib borib qo‘yib A_n^2 ning qiymatini topamiz.

$$A_n^2=n\left(a^2+a(n-1)d+\frac{d^2}{6}\left(2n^2-3n+1\right)\right) \qquad (4)$$

Demak, 2-masalaning yechimi (4) ko‘rinishda ekan.

Xususiy holda

$$\sum_{k=1}^{n}(2k-1)^2=1^2+3^2+\dots+(2n-1)^2=n\frac{4n^2-1}{3} \qquad (5)$$

$$\sum_{k=1}^{n}(3k-2)^2=1^2+4^2+\dots+(3n-2)^2=n\frac{6n^2-3n-1}{2}. \qquad (6)$$

Endi yig‘indilarni hisoblashning 2-usulini keltiramiz.

Buning uchun bizga kerak bo‘ladigan ba’zi soda yig‘indilarning formulalarini yozib olamiz.

1. $\sum_{k=1}^{n} k = \dfrac{n(n+1)}{2}$. (7)

2. $\sum_{k=1}^{n} k^2 = \dfrac{n(n+1)(2n+1)}{6}$. (8)

3. $\sum_{k=1}^{n} k^3 = \dfrac{n^2(n+1)^2}{4}$. (9)

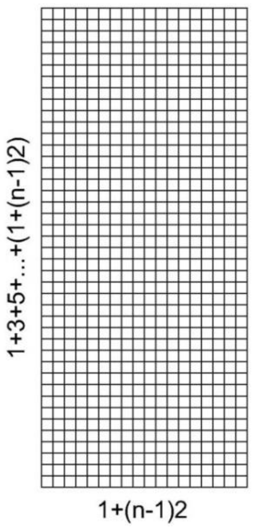
Bu usulni to‘g‘ri to‘rtburchak usuli deyishimiz mumkin. Ya’ni, geometriyaga murojat etib, geometrik shakldan hamda tushunchalardan foydalangan holda algebraik yig‘indilarning formulalarini keltirib chiqaramiz. Shunu aytib o‘tish joyizki, bu geometrik usuldan X-XI asrlarda buyuk Eron matematigi Al-Karaji ba’zi sodda yig‘indilarni hisoblashda foydalanilgan [1]. Bu usul haqida oson tushuncha hosil qilishimiz uchun soddaroq yig‘indilarni hisoblashdan boshlaymiz. Buning uchun yuqoridagi (5) yig‘indini hisoblaymiz.

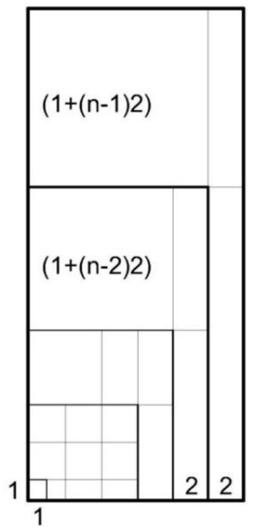
4-misol. $S_n^2 = \sum_{k=1}^{n}(2k-1)^2 = 1^2 + 3^2 + \ldots + (1+(n-1)2)^2$ ni hisoblang.

Bu misol birinchi usul bilan o‘z isbotini topgan edi. Endi bu usul bilan ham isbotlab natijalarni taqqoslaymiz.

Bu yig‘indini hisoblash uchun bo‘yi $(1+3+5+\ldots+(1+(n-1)2)$ ta satrdan, enida esa bo‘yini tashkil qilgan yig‘indini 1 dan boshqa har bir qo‘shiluvchisiga 2 tadan ustunni va 1 uchun esa 1 ta ustunni mos qilib olib jami $(1+(n-1)2)$ ta ustundan iborat to‘g‘ri to‘rtburchak olamiz (1-rasm). Satr va ustunlarning kesishishidan to‘rtburchakning ichida kvadratchalar hosil bo‘ladi. Jami kvadratchalar soni ustun va satrdagi kvadatchalarning ko‘paytmasiga teng. Ya’ni,

$$(1+3+5+\ldots+(n-1)2)\cdot(1+(n-1)2) = \frac{2+(n-1)2}{2}n(2n-1) = n^2(2n-1). \quad (10)$$





1-rasm 2-rasm

Endi to'rtburchak ichidan $1+(n-1)2$ ta satrni va 2 ta ustunni 2-rasmdagi ko'rinishda ajratib olamiz va ajratib olgan sohamizdagi jami kavadratchalar sonini hisoblaymiz.

$$(1+(n-1)2)^2+2(1+3+5+...+(1+(n-2)2)=(1+(n-1)2)^2+$$
$$+2\left(\frac{1+(1+(n-2)2)}{2}(n-1)\right)=(1+(n-1)2)^2+2(n-1)^2$$

Qaralayotgan sohada $(1+(n-1)2)^2+2(n-1)^2$ ta kvadratcha bor ekan. Endi to'rtburchakning qolgan qismidan yana shu ko'rinishda $1+(n-2)2$ ta satrni va 2 ta ustunni ajratib olib, undagi kvadratchalar sonini hisoblaymiz.

$$(1+(n-2)2)^2+2(1+3+5+...+(1+(n-3)2)=(1+(n-2)2)^2+$$
$$+2\left(\frac{1+(1+(n-3)2)}{2}(n-2)\right)=(1+(n-2)2)^2+2(n-2)^2$$

Xuddi shu ko'rinishda davom etib, yani to'rtburchagimizning bo'yini tashkil etuvchi yig'indining har bir qo'shiluvchisiga mos alohida sohalarni ajratib olib (2-rasm), undagi kvadratchalar sonlarini ketma ket qilib yozib olamiz.

$$(1+(n-1)2)^2+2(n-1)^2,\ (1+(n-2)2)^2+2(n-2)^2,......,3^2,1^2$$

Bu qiymatlarning barchasini yig'indisi to'g'ri to'rtburchakning ichidagi jami kvadratchalar soniga (10)ga teng.

$$\left(1^2+3^2...+(1+(n-2)2)^2+(1+(n-1)2)^2\right)+2\left(1^2+2^2+...+(n-2)^2+(n-1)^2\right)=n^2(2n-1)$$

Endi (8) dan foydalanib, $S_n^2+2\frac{n(n-1)(2n-1)}{6}=n^2(2n-1)$ tenglikni olamiz. Bu yerdan S_n^2 ni topamiz.

$$S_n^2=\frac{3n^2(2n-1)}{3}-\frac{n(n-1)(2n-1)}{3}=\frac{n(4n^2-1)}{3}$$

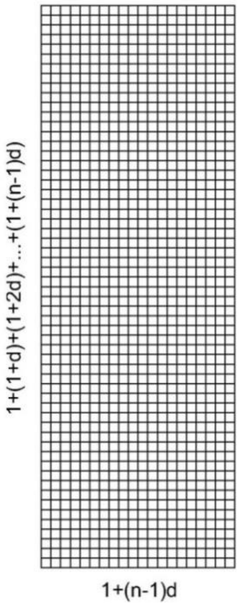
Demak, $S_n^2=\sum_{k=1}^{n}(2k-1)^2=\frac{n(4n^2-1)}{3}$ ekan va u (5) natija bilan bir xil.

Endi masalani umumiylashtirsak.

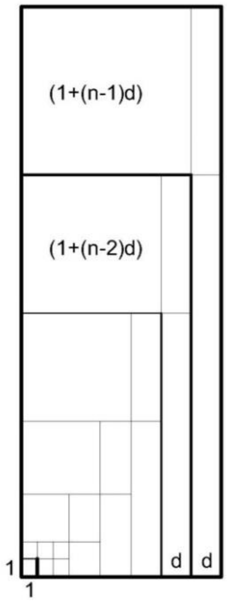
5-misol. $D_n^2=\sum_{k=1}^{n}(1+(n-1)d)^2=1^2+(1+d)^2+...+(1+(n-1)d)^2$ yig'indini hisoblang.

Bu misolda ham xuddi yuqoridagi misolga o'xshash to'g'ri to'rtburchak olamiz. To'rtburchakning bo'yi $(1+(1+d)+...+(1+(n-1)d)$ ta satrdan eni esa $(1+(n-1)d)$ ta ustundan iborat bo'lsin (3-rasm). Undagi jami kvadratchalar sonini hisoblaymiz.

$$(1+(1+d)+...+(1+(n-1)d)(1+(n-1)d)=\frac{2+(n-1)d}{2}n(1+(n-1)d) \qquad (11)$$



3-rasm



4-rasm

Endi to‘rtburchakning bo‘yidan $(1+(n-1)d)$ ta satrni enidan d ta ustunni ajratib olamiz (4-rasm). Bu sohadagi kvadratchalar sonini hisoblaymiz.

$$(1+(n-1)d)^2+d\left(1+(1+d)+...(1+(n-2)d)\right)=(1+(n-1)d)^2+d\left(\frac{2+(n-2)d}{2}(n-1)\right)=$$

$$=(1+(n-1)d)^2+d\left(\frac{2-d+(n-1)d}{2}(n-1)\right)=(1+(n-1)d)^2+\frac{2d-d^2}{2}(n-1)+\frac{d^2}{2}(n-1)^2$$

To‘rtburchakning ajratib olgan sohamizdan tashqaridagi qismida ham shu ishni takrorlaymiz. Bu safar $(1+(n-2)d)$ ta satrni va unga mos d ta ustunni olamiz va undagi kvadratchalar sonini ham hisoblaymiz (4-rasm).

$$(1+(n-2)d)^2+d\left(1+(1+d)+...(1+(n-3)d)\right)=(1+(n-2)d)^2+d\left(\frac{2+(n-3)d}{2}(n-2)\right)=$$

$$=(1+(n-2)d)^2+d\left(\frac{2-d+(n-2)d}{2}(n-2)\right)=(1+(n-2)d)^2+\frac{2d-d^2}{2}(n-2)+\frac{d^2}{2}(n-2)^2$$

Shu ko‘rinishda oxirigacha davom etib, olingan natijalarni barchasini yozib olamiz.

$$(1+(n-1)d)^2+\frac{2d-d^2}{2}(n-1)+\frac{d^2}{2}(n-1)^2,$$

$$(1+(n-2)d)^2+\frac{2d-d^2}{2}(n-2)+\frac{d^2}{2}(n-2)^2$$

....

$$(1+(n-2)d)^2+\frac{2d-d^2}{2}1+\frac{d^2}{2}1^2$$

1^2

Hosil bo'lgan qiymatlar ketma ketligining barchasini yig'indisi to'rtburchakning ichidagi jami kvadratchalar soni (11) ga teng.

$$(1^2+(1+d)^2+...+(1+(n-1)d)^2+\frac{2d-d^2}{2}(1+2+...(n-1))+\frac{d^2}{2}(1^2+2^2+...(n-1)^2)=$$
$$=\frac{2+(n-1)d}{2}n(1+(n-1)d)$$

(7) va (8) yig'indilarning formulalaridan foydalanib, yuqoridagi ifodani quydagicha yozib olamiz.

$$D_n^2+\frac{2d-d^2}{2}\frac{n(n-1)}{2}+\frac{d^2}{2}\frac{n(n-1)(2n-1)}{6}=\frac{2+(n-1)d}{2}n(1+(n-1)d)$$

Bu yerdan D_n^2 ni topib olamiz.

$$D_n^2=n+\frac{dn(n-1)(2dn+6-d)}{6} \qquad (12)$$

Demak, bu (12) formula yordamida birinchi hadi 1 ga, ayirmasi ixtiyoriy musbat butun d ga teng bo'lgan arifmetik progressiyaning hadlarining kvadratlaridan tuzilgan qator yig'indisini topish mumkin. (6) yig'indini shu formula bo'yicha hisoblab ko'ramiz. Buning uchun $d=3$ ni (12) ga qo'yamiz.

$$\sum_{k=1}^{n}(3k-2)^2=n+\frac{3n(n-1)(2\cdot 3n+6-3)}{6}=n+\frac{n(n-1)(6n+3)}{2}=n\frac{6n^2-3n-1}{2}$$

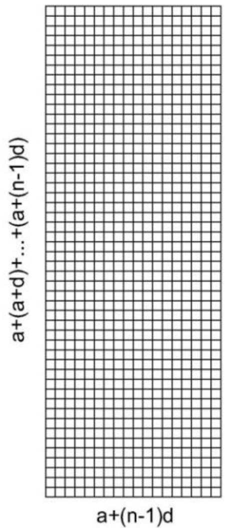
Nihoyat (6) natijani yana bir bor isbotladik.

Hozir biz yuqoridagi ikkita misolda birinchi hadi 1^2 bo'lgan yig'indilarni hisobladik. Agar shu birinchi hadni boshqa bir musbat butun sonni kvadrati bilan almashtirsak misol qiyinlashishi bilan bir qatorda yanada umumlashadi. Axir hamma arifmetik progressiyalar ham 1 sonidan boshlanavermaydiku. Shu hisobda quyidagicha misolni ko'ramiz.

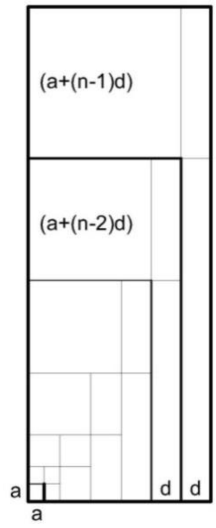
6-misol. $A_n^2=\sum_{k=1}^{n}(a+(n-1)d)^2=a^2+(a+d)^2+...+(a+(n-1)d)^2$ yig'indini hisoblang.

Bu misolda bo'yi $(a+(a+d)+...+(a+(n-1)d)$ satrdan, eni esa $(a+(n-1)d)$ ustundan iborat to'g'ri to'rtburchak olamiz. Yani satrlar sonini ifodalovchi yig'indining a qo'shiluvchisidan tashqari boshqa har bir qo'shiluvchisiga d tadan ustun va a uchun esa a ta ustun olgan holda jami ustunlar soni $(a+(n-1)d)$ ta bo'ladi (5-rasm). Dastlab to'g'ri to'rtburchakning ichidagi jami kvadratchalar sonini topib olamiz.

$$(a+(a+d)+...+(a+(n-1)d)(a+(n-1)d)=\left(\frac{2a+(n-1)d}{2}n\right)(a+(n-1)d) \qquad (13)$$



5-rasm



6-rasm

Bu misolda ham yuqoridagi misollar kabi to‘rtburchagimizdan sohalar ajratib olib, undagi kvadratchalar sonini hisblaymiz. Birinchi sohamiz uchun $(a+(n-1)d)$ ta satrni va d ta ustunni tanlab olamiz(6-rasm).

$$\left(a+(n-1)d\right)^2+d\left(a+(a+d)+...(a+(n-2)d)\right)=\left(a+(n-1)d\right)^2+d\left(\frac{2a+(n-2)d}{2}(n-1)\right)=$$

$$=\left(a+(n-1)d\right)^2+d\left(\frac{2a-d+(n-1)d}{2}(n-1)\right)=\left(a+(n-1)d\right)^2+\frac{2ad-d^2}{2}(n-1)+\frac{d^2}{2}(n-1)^2$$

Bu ifoda tanlagan birinchi sohamizdagi kvadratchalar sonini ifoda etadi.

Keyingi qadamda $(a+(n-2)d)$ ta satrni va yana d ta ustunni tanlab ajratib olamiz. Endi undagi kvadratchalar sonini topamiz(6-rasm).

$$\left(a+(n-2)d\right)^2+d\left(a+(a+d)+...(a+(n-3)d)\right)=\left(a+(n-2)d\right)^2+d\left(\frac{2a+(n-3)d}{2}(n-2)\right)=$$

$$=\left(a+(n-2)d\right)^2+d\left(\frac{2a-d+(n-2)d}{2}(n-2)\right)=\left(a+(n-2)d\right)^2+\frac{2ad-d^2}{2}(n-2)+\frac{d^2}{2}(n-2)^2$$

Shu zayilda hisoblashlarni davom ettirdigan bo‘lsak har safar yuqoridagiga o‘xshash natijalarni olaveramiz. Olgan qiymatlarimizni yozib olamiz.

$$\left(a+(n-1)d\right)^2+\frac{2ad-d^2}{2}(n-1)+\frac{d^2}{2}(n-1)^2$$

$$\left(a+(n-2)d\right)^2+\frac{2ad-d^2}{2}(n-2)+\frac{d^2}{2}(n-2)^2$$

……………………………………..

$$(a+d)^2+\frac{2ad-d^2}{2}1+\frac{d^2}{2}1^2$$

$$a^2$$

Bu qiymatlarning yig‘indisi 5-rasmdagi to‘rtburchakning ichidagi jami kvadratchalar soniga (13)ga teng.

$$(a^2+(a+d)^2+...+(a+(n-1)d)^2+\frac{2ad-d^2}{2}(1+2+...(n-1))+\frac{d^2}{2}(1^2+2^2+...(n-1)^2)=$$

$$=\left(\frac{2a+(n-1)d}{2}n\right)(a+(n-1)d)$$

$$A_n^2+\frac{2ad-d^2}{2}\frac{n(n-1)}{2}+\frac{d^2}{2}\frac{n(n-1)(2n-1)}{6}=\left(\frac{2a+(n-1)d}{2}n\right)(a+(n-1)d)$$

Bu yerdan A_n^2 ni topib olamiz.

$$A_n^2=\frac{d^2}{3}n^3+\left(ad-\frac{d^2}{2}\right)n^2+\left(a^2-ad+\frac{d^2}{6}\right)n \qquad (14)$$

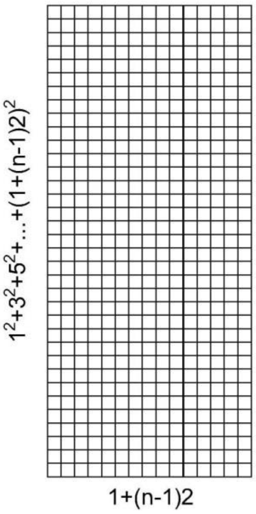
Topilgan (14) ifoda (4) ifoda bilan bir xil ma’no anglatadi faqat boshqacha ko‘rinishi. (14) formula (5) , (6) kabi yig‘indilarning istalgan qism yig‘indisini ham topish imkonini beradi. Shu bilan arifmetik progressiyani tashkil etuvchi sonlarning kvadratlari yig’indisini topish masalasiga yakun yasasak ham bo‘ladi.

Endi arifmetik progressiya hadlarining kublaridan tuzilgan yig‘indilarni topish masalasi bilan shug‘illanamiz.

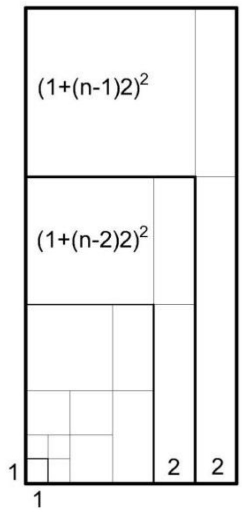
7-misol. $\sum_{k=1}^{n}(2k-1)^3=1^3+3^3+...+(1+(n-1)2)^3$ yig‘indini hisoblang.

Bu misol uchun bo‘yi $(1^2+3^2+...+(1+(n-1)2)^2)$ ta satrdan, eni esa $(1+(n-1)2)$ ustundan iborat to‘g‘ri to‘rtburchak olamiz(7-rasm). Satrlar va ustunlarning kesishishidan hosil bo‘lgan jami kvadratchalar sonini (12) dan foydalangan holda topamiz.

$$(1^2+3^2+...+(1+(n-1)2)^2)(1+(n-1)2)=\frac{n^2(n+1)^2}{4}(1+(n-1)2) \qquad (15)$$



7-rasm



8-rasm

Endi to‘rtburchakning bo‘yidan $(1+(n-1)2)^2$ ta satrni va enidan 2 ta ustunni 8-rasmdagidek qilib ajratib olamiz. Ajratib olgan sohamizdagi kvadratchalar sonini

$\left(1+\left(n-1\right)2\right)^{3}+2(1^{2}+3^{2}+...+(1+\left(n-2\right)2)^{2})$ ga teng. (5) formuladan foydalansak bu ifodamiz

$$\left(1+\left(n-1\right)2\right)^{3}+2\left(n-1\right)\frac{4\left(n-1\right)^{2}-1}{3}=\left(1+\left(n-1\right)2\right)^{3}+\frac{8}{3}(n-1)^{3}-\frac{2}{3}(n-1)$$

ko‘rinishga keladi. Demak bu sohada $\left(1+\left(n-1\right)2\right)^{3}+\frac{8}{3}(n-1)^{3}-\frac{2}{3}(n-1)$ ta kvadratcha mavjud.

To‘rtburchakning qolgan qismining bo‘yidan $\left(1+\left(n-2\right)2\right)^{2}$ satrni, enidan 2 ta ustunni tanlab olib (8-rasm), undagi kvadratchalar sonini xuddi yuqoridagi kabi hisoblasak, $\left(1+\left(n-2\right)2\right)^{3}+\frac{8}{3}(n-2)^{3}-\frac{2}{3}(n-2)$ ga teng ekanligi kelib chiqadi. Shu ko‘rinishda davom etib barcha sohalardagi kvadratchalar sonini ketma ket yozib olamiz.

$$\left(1+\left(n-1\right)2\right)^{3}+\frac{8}{3}(n-1)^{3}-\frac{2}{3}(n-1)$$

$$\left(1+\left(n-2\right)2\right)^{3}+\frac{8}{3}(n-2)^{3}-\frac{2}{3}(n-2)$$

…………………………………

$$3^{3}+\frac{8}{3}1^{3}-\frac{2}{3}1$$

$$1^{3}$$

Bu qiymatlarning barchasini yig‘indisi bizga 7-rasmdagi jami kvadratchalar soni (15) ni beradi.

$$(1^{3}+3^{3}+...+(1+\left(n-1\right)2)^{3}+\frac{8}{3}\left(1^{3}+2^{3}+...+\left(n-1\right)^{3}\right)-\frac{2}{3}\left(1+2+...\left(n-1\right)\right)=$$

$$=\frac{n^{2}(n+1)^{2}}{4}\left(1+\left(n-1\right)2\right)$$

Bu ifodani, (7) va (9) formulalardan foydalanib quydagicha yozib olamiz.

$$\sum_{k=1}^{n}(2k-1)^{3}+\frac{8}{3}\cdot\frac{n^{2}\left(n-1\right)^{2}}{4}-\frac{2}{3}\cdot\frac{n(n-1)}{2}=\frac{n^{2}(n+1)^{2}}{4}\left(1+\left(n-1\right)2\right)$$

Bu yerdan $\sum_{k=1}^{n}(2k-1)^{3}$ ni topib olishimiz mumkin. Demak, $\sum_{k=1}^{n}(2k-1)^{3}=n^{2}(2n^{2}-1)$ ekan.

Endi masalani umumiy holda ko‘ramiz.

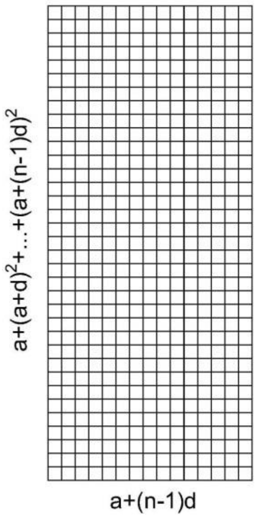
8-misol. $A_{n}^{3}=\sum_{k=1}^{n}(a+\left(n-1\right)d)^{3}=a^{3}+(a+d)^{3}+...+(a+\left(n-1\right)d)^{3}$ yig‘indini hisoblang.

Yig‘indining qiymatini hisoblab beruvchi formulani to‘g‘ri to‘rtburchak usulidan foydalanib keltirib chiqaramiz.

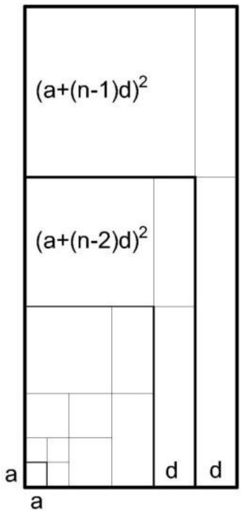
Buning uchun bo‘yi $\left(a^2+(a+d)^2+...+(a+(n-1)d)^2\right)$ ta satrdan iborat, eni esa $(a+(n-1)d)$ ta ustundan iborat to‘g‘ri to‘rtburchak olamiz (9-rasm). To‘rtburchakning ichida hosil bo‘lgan jami kvadartchalarning jami soni $(a^2+(a+d)^2+...+(a+(n-1)d)^2)(a+(n-1)d)$ ga teng. Yuqoridagi (14) formuladan foydalanib bu ifodani quyidagicha yozib olamiz.

$$\left(\frac{d^2}{3}n^3+\left(ad-\frac{d^2}{2}\right)n^2+\left(a^2-ad+\frac{d^2}{6}\right)n\right)(a+(n-1)d) \qquad (16)$$

Endi to‘tburchak bo‘yidan $(a+(n-1)d)^2$ ta satrni enidan esa d ta ustunni ajratib olamiz (10-rasm). Ajratib olgan sohamizdagi kvadratchalar sonini hisoblaymiz.



9-rasm



10-rasm

$$(a+(n-1)d)^3+d(a^2+(a+d)^2+...+(a+(n-2)d)^2)=$$

$$=(a+(n-1)d)^3+d\left(\frac{d^2}{3}(n-1)^3+\left(ad-\frac{d^2}{2}\right)(n-1)^2+\left(a^2-ad+\frac{d^2}{6}\right)(n-1)\right)=$$

$$=(a+(n-1)d)^3+\frac{d^3}{3}(n-1)^3+\left(ad^2-\frac{d^3}{2}\right)(n-1)^2+\left(a^2d-ad^2+\frac{d^3}{6}\right)(n-1)$$

Demak bu sohada jami kvadratchalar soni

$$(a+(n-1)d)^3+\frac{d^3}{3}(n-1)^3+\left(ad^2-\frac{d^3}{2}\right)(n-1)^2+\left(a^2d-ad^2+\frac{d^3}{6}\right)(n-1)$$

ga teng.

To‘rtburchakning qolgan qismidan yana yuqoridagiga o‘xshab bo‘yidan $(a+(n-2)d)^2$ ta satrni, enidan d ta ustunni ajratib olamiz va undagi kvadratchalar sonini hisoblasak,

$$(a+(n-2)d)^3+\frac{d^3}{3}(n-2)^3+\left(ad^2-\frac{d^3}{2}\right)(n-2)^2+\left(a^2d-ad^2+\frac{d^3}{6}\right)(n-2)$$

koʻrinishdagi natijani olamiz.

Shu jarayonni oxirigacha davom ettirib, yani toʻrtburchakning boʻyini ifodalovchi yigʻindining a^2 dan boshqa har bir qoʻshiluvchisiga d ta ustunni mos qoʻyib hosil boʻlgan sohalardagi kvadratchalar sonini hisoblab ketma ket yozib olamiz. Bunda a^2 ga esa a ta ustun mos qoʻyiladi (10-rasm).

$$\left(a+(n-1)d\right)^3+\frac{d^3}{3}(n-1)^3+\left(ad^2-\frac{d^3}{2}\right)(n-1)^2+\left(a^2d-ad^2+\frac{d^3}{6}\right)(n-1)$$

$$\left(a+(n-2)d\right)^3+\frac{d^3}{3}(n-2)^3+\left(ad^2-\frac{d^3}{2}\right)(n-2)^2+\left(a^2d-ad^2+\frac{d^3}{6}\right)(n-2)$$

…………………………..

$$\left(a+d\right)^3+\frac{d^3}{3}1^3+\left(ad^2-\frac{d^3}{2}\right)1^2+\left(a^2d-ad^2+\frac{d^3}{6}\right)1$$

$$a^3$$

Hosil boʻlgan qiymatlar ketma ketligining yigʻindisi toʻrtburchak ichidagi jami kvadratchalar soni (16) ga teng.

$$(a^3+\left(a+d\right)^3+...+\left(a+(n-1)d\right)^3)+\frac{d^3}{3}\left(1^3+2^3+...+(n-1)^3\right)+$$

$$+\left(ad^2-\frac{d^3}{2}\right)\left(1^2+2^2+...+(n-1)^2\right)+\left(a^2d-ad^2+\frac{d^3}{6}\right)(1+2+...+(n-1)=$$

$$=\left(\frac{d^2}{3}n^3+\left(ad-\frac{d^2}{2}\right)n^2+\left(a^2-ad+\frac{d^2}{6}\right)n\right)\left(a+(n-1)d\right)$$

Endi (7), (8) va (9) formulalardan foydalanib, yuqoridagi ifodani quyidagicha yozib olamiz.

$$A_n^3+\frac{d^3}{3}\cdot\frac{n^2(n-1)^2}{4}+\left(ad^2-\frac{d^3}{2}\right)\cdot\frac{n(n-1)(2n-1)}{6}+\left(a^2d-ad^2+\frac{d^3}{6}\right)\cdot\frac{n(n-1)}{2}=$$

$$=\left(\frac{d^2}{3}n^3+\left(ad-\frac{d^2}{2}\right)n^2+\left(a^2-ad+\frac{d^2}{6}\right)n\right)\left(a+(n-1)d\right)$$

Algebraik soddalashtirish ishlarini bajarib, A_n^3 ni topib olamiz.

$$A_n^3=\frac{d^3}{4}n^4+\left(ad^2-\frac{d^3}{2}\right)n^3+\left(\frac{3a^2d}{2}-\frac{3ad^2}{2}+\frac{d^3}{4}\right)n^2+\left(a^3-\frac{3a^2d}{2}+\frac{ad^2}{2}\right)n$$

Demak,

$$A_n^3=\sum_{k=1}^{n}(a+(n-1)d)^3=\frac{d^3}{4}n^4+\left(ad^2-\frac{d^3}{2}\right)n^3+\left(\frac{3a^2d}{2}-\frac{3ad^2}{2}+\frac{d^3}{4}\right)n^2+$$

$$+\left(a^3-\frac{3a^2d}{2}+\frac{ad^2}{2}\right)n$$

ekanligini topdik.

To‘g‘ri to‘rtburchak usuli yordamida hisoblangan yig‘indilardan shuni xulosa qilib aytish mumkinki bu usul yordamida istalgan A_n^m $(m \in N)$ larni topish mumkin. Buning uchun to‘g‘ri to‘rtburchakning bo‘yini A_n^{m-1} ta satrdan, enini esa $(a+(n-1)d)$ ta ustundan iborat qilib olish yetarli. (Bu yerda a va d ixtiyoriy musbat butun son) Hisoblash jarayoni esa yuqoridagi misollarniki kabi amalga oshiriladi.

Xulosada keltirilgan fikrga asoslanib,

$$A_n^1 = \sum_{k=1}^{n}\left(a+\left(k-1\right)d\right) = a+\left(a+d\right)+...+(a+\left(n-1\right)d)$$

ni yani, bizga tanish bo‘lagan arifmetik progressiyaning hadlari yig‘indisini shu usul yordamida $\frac{2a+\left(n-1\right)d}{2}n$ ga teng ekanligini chiroyli ko‘rinishda isbotlashimiz mumkin. Yig‘indilar formulalarining bu usuldagi isbotlanishi o‘quvchilarning tasavvurini oshirish bilan bir qatorda ularga qiziqarli bo‘lishi turgan gap.

FOYDALANILGAN ADABIYOTLAR:

1. Daan-Dalmediko A., Peiffer J. Ways and labyrinths: essays on the history of mathematics. - M .: Mir, 1986.- 433 p.

2.*Окунев Л. Я. Комбинаторные задачи* на *шахматной доске*.- *М*.: *Л.* : *ОНТИ*, *1935*. - 88 с.

3.Маматов М.Ш., Темуров С.Ю., Махмудова Д.М., Куницын А.З. О применении информационно –коммуникационных технологий при развитии самостоятельного творческого мышления молодежи// Актуальные проблемы гуманитарных и естественных наук. Москва. – 2012. – №3. – С. 234 – 241.